

Kinetic Theory of Response Functions for the Hard Sphere Granular Fluid

Aparna Baskaran

Physics Department, Syracuse University, Syracuse, NY 13244

James W. Dufty

Department of Physics, University of Florida, Gainesville, FL 32611

J. Javier Brey

Física Teórica, Universidad de Sevilla,

Apartado de Correos 1065, E-41080, Sevilla, Spain

(Dated: February 1, 2008)

Abstract

The response functions for small spatial perturbations of a homogeneous granular fluid have been described recently. In appropriate dimensionless variables, they have the form of stationary state time correlation functions. Here, these functions are expressed in terms of reduced single particle functions that are expected to obey a linear kinetic equation. The functional assumption required for such a kinetic equation, and a Markov approximation for its implementation are discussed. If, in addition, static velocity correlations are neglected, a granular fluid version of the linearized Enskog kinetic theory is obtained. The derivation makes no *a priori* limitation on the density, space and time scale, nor degree of inelasticity. As an illustration, recently derived Helfand and Green-Kubo expressions for the Navier-Stokes order transport coefficients are evaluated with this kinetic theory. The results are in agreement with those obtained from the Chapman-Enskog solution to the nonlinear Enskog kinetic equation.

PACS numbers: 05.20.Dd, 45.70.-n, 05.60.-k, 47.10.ab

I. INTRODUCTION

One of the most productive methods to study transport in normal fluids is through the measurement, simulation, and theory of linear response functions [1–4]. Of particular interest are those that describe the linear response of the “hydrodynamic fields” (mass, energy, and momentum densities) to small spatial perturbations of the homogeneous equilibrium state. The terminology, hydrodynamic fields, is due to the fact that these are the variables expected to obey the phenomenological hydrodynamic equations on large space and time scales. The response functions provide the means to study such hydrodynamic excitations starting from their fundamental basis in non-equilibrium statistical mechanics. For example, they provide the formally exact Helfand [5] and Green-Kubo [4, 6] representations for transport coefficients. More generally, the response functions describe the broader range of excitations on shorter space and time scales as well. One of the most instructive theoretical approaches to their evaluation has been kinetic theory, with the greatest progress made for the idealized fluid of hard spheres [4, 7].

Recently, this linear response approach has been extended to granular fluids [8, 9]. The objective here is to demonstrate the application of kinetic theory methods for the evaluation of the granular response functions. Only the case of smooth, inelastic hard spheres is considered both for simplicity and to parallel closely the corresponding developments for normal fluids. Such an idealized model still captures the most important features of many granular fluids [10]. The usual notion of kinetic theory is a nonlinear equation for the probability density in single particle phase space. An advantage of the linear response functions is that their kinetic equation is inherently linear. For practical purposes, a Markovian approximation to this linear kinetic equation is described, based on the neglect of dynamical correlations. The approximations leading to this equation, and the differences between its implications for normal and granular fluids, are discussed. The nature of the approximation does not *a priori* assume weak dissipation, low density, or large length and time scales. In the elastic limit, it becomes the linear Enskog kinetic equation for the response functions of a normal equilibrium fluid. A granular Enskog limit is described here as well, by the further neglect of all velocity correlations in the Markov approximation.

A related set of time correlation functions involving the fluxes of the hydrodynamic fields, instead of the fields themselves, determine the Helfand and Green-Kubo representations

for the transport coefficients of the Navier-Stokes order hydrodynamic equations [8, 9]. These can be evaluated by the same kinetic theory developed for the response functions. This is described in detail for the shear viscosity in the Markov approximation, indicating how the time dependence of these flux correlation functions can be determined, as well as the associated transport coefficient. The remaining Navier-Stokes transport coefficients are evaluated in the Appendix D. It is confirmed that, in the linear Enskog approximation, the results agree in detail with those obtained from the Chapman-Enskog method to solve the nonlinear Enskog equation for the distribution function [11].

A similar program has already been carried out in the limited context of dynamics for an impurity particle in a granular fluid [12, 13]. In that case, the only hydrodynamic field is the impurity particle probability density, and the response function is its autocorrelation function. The hydrodynamic equation is a diffusion equation, and the Green-Kubo expression for the diffusion coefficient is given by the time integral of the velocity autocorrelation function. This was evaluated by kinetic theory in the Enskog approximation, and both the correlation function and the transport coefficient were compared with molecular dynamics simulation data over a wide range of densities and degrees of inelasticity. The results provide an instructive characterization of the domain of validity for the Enskog (Enskog-Lorentz in this case) kinetic equation, and expose important differences between impurity dynamics in normal and granular fluids. The presentation here constitutes an extension of that theoretical analysis to the full range of multi-particle mass, energy, and momentum transport.

The origin of a kinetic theory is an exact hierarchy (the BBGKY hierarchy [4, 7]) of equations for the reduced few particle representations for a property of interest. A kinetic equation is comprised of the first hierarchy equation together with a “closure”, expressing the solution to the second hierarchy equation as a functional of that for the first. This leads to a closed, deterministic equation for the latter which is the kinetic equation. Practical methods have been developed for normal fluids based on inversion of cluster expansions and partial resummations, as well as more phenomenological estimates. This is the point at which kinetic theory confronts the difficult many body problem, and one objective of the current work is to motivate a corresponding attention to such details for the granular fluid. Only in this way can the qualitative speculations about differences between normal and granular fluid be made more precise. An example of such uncertainties is the role of velocity

correlations in the construction of the closure. It is well known that velocity correlations generated by collective many particle collisions (e.g., ring collisions) are responsible for the dominant density dependence of transport coefficients at very high densities, but they are relatively unimportant at low to moderate densities where simple spatial correlations (e.g., excluded volume effects) are dominant. The latter are incorporated in the Enskog approximation for accurate corrections to the predictions of the low density Boltzmann equation. Granular fluids introduce a complication to this separation of dynamical velocity correlations and static structural correlations, according to the density considered. In these systems, there are inherent static velocity correlations, not directly associated with many particle dynamics, that are present even at low to moderate densities. The kinetic theory for response functions provides an appropriate setting for the study of the quantitative importance of these correlations on properties of interest.

The response functions in the Markovian approximation are expressed in terms of the linear generator for dynamics. Questions about the existence and dominance of hydrodynamics can be made precise at this point, by asking if the hydrodynamic modes (eigenvalues) appear in the spectrum of this operator and if they are the slowest modes. The first part, existence of hydrodynamic modes, can be demonstrated at long wavelengths if expected conditions of analyticity are satisfied (see Sec. VI below). The second issue of dominance at long times can be addressed practically using simplified kinetic models for this generator. Such results support the primary assumptions in references [8] and [9], for a derivation of formally exact expressions for the hydrodynamic transport coefficients in terms of time correlation functions. However, the linear kinetic equation applies as well to short space and time scales, that are important for non-hydrodynamic response.

The main points of this analysis are summarized in the last section.

II. LINEAR RESPONSE FUNCTIONS

An idealized granular fluid of N smooth, inelastic hard spheres ($d = 3$) or disks ($d = 2$) of mass m and diameter σ is considered. The inelasticity is characterized by a constant coefficient of normal restitution α . The properties of interest are the average number density $n(\mathbf{r}, t)$, the granular temperature $T(\mathbf{r}, t)$, and the local flow velocity $\mathbf{U}(\mathbf{r}, t)$, collectively denoted by $y(\mathbf{r}, t) \equiv \{y_\beta(\mathbf{r}, t)\}$. The response to be studied here is the variation of these

fields at time t due to a variation in their initial values at time $t = 0$. The initial conditions are spatial variations of a homogeneous reference state, $y_\beta(\mathbf{r}, 0) = y_{\beta,h}(0) + \delta y_\beta(\mathbf{r}, 0)$. In the most general case, the response $\delta y_\beta[\mathbf{r}, t|y(0)]$ depends nonlinearly on the perturbations $\delta y(\mathbf{r}, 0)$, but if these are small, the linear order gives the dominant response,

$$\delta y_\beta[\mathbf{r}, t|y(0)] = \sum_\gamma \int d\mathbf{r}' C_{\beta\gamma}(\mathbf{r} - \mathbf{r}', t) \delta y_\gamma(\mathbf{r}', 0), \quad (1)$$

with the linear response functions defined as

$$C_{\beta\gamma}(\mathbf{r} - \mathbf{r}', t) = \left[\frac{\delta y_\beta[\mathbf{r}, t|y(0)]}{\delta y_\gamma(\mathbf{r}', 0)} \right]_{\delta y(0)=0}. \quad (2)$$

Since the reference state is homogeneous, the linear response functions depend on \mathbf{r} and \mathbf{r}' only through their difference.

The difference between granular and normal fluids occurs already at the level of the homogeneous reference state. For normal fluids, this is the equilibrium stationary state and all the time dependence of the response functions is due to the spatial perturbations. For granular fluids, the homogeneous reference state is inherently time-dependent, even without perturbation, due to the “cooling” of inelastic collisions. As a consequence, the temperature T_h of the homogeneous granular fluid decreases in time according with a cooling law

$$\frac{\partial T_h(t)}{\partial t} = -\zeta_0[T_h(t)] T_h(t), \quad (3)$$

where the cooling rate $\zeta_0(T_h)$ is a characteristic function of the homogeneous cooling reference state. Thus the relevant response at time t is measured relative to the reference homogeneous state at the same time rather than to the initial state. Then, dimensionless fields δy_β^* are introduced by

$$\{\delta y_\beta^*\} \equiv \left\{ \frac{\delta y_\beta}{\bar{y}_{\beta,h}(t)} \right\} \equiv \left\{ \frac{\delta n}{n_h}, \frac{\delta T}{T_h(t)}, \frac{\delta \mathbf{U}}{v_0(t)} \right\}, \quad (4)$$

where the definition of the reference fields $\bar{y}_{\beta,h}(t)$ follows from the second identity and

$$v_0(t) \equiv \left[\frac{2T_h(t)}{m} \right]^{1/2} \quad (5)$$

is a thermal velocity. The dependence of the linear response functions on $\mathbf{r} - \mathbf{r}'$ suggests the utility of a Fourier representation of Eq. (1) that is expressed in the form

$$\delta \tilde{y}_\beta^*(\mathbf{k}^*, s) = \sum_\gamma \tilde{C}_{\beta\gamma}^*(\mathbf{k}^*, s) \delta \tilde{y}_\gamma^*(\mathbf{k}^*, 0), \quad (6)$$

where a tilde over a function denotes its dimensionless Fourier transform, defined by

$$\begin{aligned}\tilde{f}(\mathbf{k}^*) &\equiv \ell^{-d} \int d\mathbf{r} e^{i\mathbf{k}^* \cdot \mathbf{r}/\ell} f(\mathbf{r}) \\ &= \int d\mathbf{r}^* e^{i\mathbf{k}^* \cdot \mathbf{r}^*} f(\mathbf{r}),\end{aligned}\tag{7}$$

with $\mathbf{r}^* = \mathbf{r}/\ell$, and dimensionless response functions have been identified as

$$\begin{aligned}\tilde{C}_{\beta\gamma}^*(\mathbf{k}^*, s) &= \bar{y}_{\beta,h}^{-1}(t) \ell^d \tilde{C}_{\beta\gamma}(\mathbf{k}^*, t) \bar{y}_{\gamma,h}(0) \\ &= \bar{y}_{\beta,h}^{-1}(t) \int d\mathbf{r} e^{i\mathbf{k}^* \cdot \mathbf{r}/\ell} C_{\beta\gamma}(\mathbf{r}, t) \bar{y}_{\gamma,h}(0).\end{aligned}\tag{8}$$

Here $\mathbf{k}^* = \mathbf{k}\ell$ is a dimensionless wavevector, with ℓ being a characteristic length of the system. Moreover, a dimensionless time scale

$$s = \int_0^t dt' \frac{v_0(t')}{\ell}\tag{9}$$

has been introduced. This time s has the interpretation of an average collision number per particle up to time t , if ℓ is chosen to be the mean free path of the particles. The remainder of this presentation focuses on the dimensionless linear response functions $\tilde{C}_{\beta\gamma}^*$.

For small \mathbf{k}^* and large s , the functions $\delta y_\beta^*(\mathbf{k}^*, s)$ are expected to obey the linearized hydrodynamic Navier-Stokes equations, and the response functions correspond to the Green functions for the solution to the initial value problem associated with those equations [9]. That description is only phenomenological, since it is parameterized by the unknown transport coefficients. A more complete and exact description for all \mathbf{k}^* and s is provided by non-equilibrium statistical mechanics, as described in references [8] and [9]. Briefly, the construction is as follows.

The hydrodynamic fields $y_\beta(\mathbf{r}, t)$ are defined from averages of the microscopic number density, energy density, and momentum density over the phase space density $\rho(\Gamma, t)$, representing the probability that the positions \mathbf{q}_r and velocities \mathbf{v}_r of the particles have specified values denoted by $\Gamma \equiv \{x_1, \dots, x_N\}$. The latter is a point in the $2Nd$ dimensional phase space with the notation $x_r \equiv \{\mathbf{q}_r, \mathbf{v}_r\}$. For any specified initial state, $\rho(\Gamma, 0)$, this probability density evolves in time according to the Liouville equation

$$\left(\frac{\partial}{\partial t} + \bar{L} \right) \rho(\Gamma, t) = 0,\tag{10}$$

with

$$\bar{L} = \sum_{r=1}^N \mathbf{v}_r \cdot \frac{\partial}{\partial \mathbf{q}_r} - \frac{1}{2} \sum_{r=1}^N \sum_{s \neq r}^N \bar{T}(x_r, x_s).\tag{11}$$

The time independent operator \bar{L} is the generator for the hard sphere dynamics, where the singular binary collisions are described by

$$\bar{T}(x_r, x_s) = \delta(q_{rs} - \sigma) |\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}| [\Theta(\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}) \alpha^{-2} b_{rs}^{-1} - \Theta(-\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs})]. \quad (12)$$

In this expression, $\mathbf{q}_{rs} = \mathbf{q}_r - \mathbf{q}_s$, $\mathbf{g}_{rs} = \mathbf{v}_r - \mathbf{v}_s$, Θ is the Heaviside step function, $\hat{\mathbf{q}}_{rs} \equiv \mathbf{q}_{rs}/q_{rs}$, and b_{rs}^{-1} is the substitution operator that replaces the velocities \mathbf{v}_r , \mathbf{v}_s by their “precollisional” values $\mathbf{v}_r'', \mathbf{v}_s''$,

$$b_{rs}^{-1} F(\mathbf{v}_r, \mathbf{v}_s) = F(\mathbf{v}_r'', \mathbf{v}_s''), \quad (13)$$

$$\mathbf{v}_r'' = \mathbf{v}_r - \frac{1 + \alpha}{2\alpha} (\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}) \hat{\mathbf{q}}_{rs}, \quad (14)$$

$$\mathbf{v}_s^{*''} = \mathbf{v}_s + \frac{1 + \alpha}{2\alpha} (\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}) \hat{\mathbf{q}}_{rs}. \quad (15)$$

For an isolated system, instead of the equilibrium state for a normal fluid, there is a spacial solution to the Liouville equation of the form

$$\rho_h(\Gamma, t) = [\ell v_0(t)]^{-Nd} \rho_h^*(\{\mathbf{q}_{rs}/\ell, \mathbf{v}_r/v_0(t)\}) \equiv [\ell v_0(t)]^{-Nd} \rho_h^*(\Gamma^*). \quad (16)$$

The dimensionless phase point $\Gamma^* \equiv \{x_1^*, \dots, x_N^*\}$ is now expressed in terms of the scaled positions and velocities $x_r^* = \{\mathbf{q}_r^*, \mathbf{v}_r^*\} \equiv \{\mathbf{q}_r/\ell, \mathbf{v}_r/v_0(t)\}$. This solution depends on the positions only through the relative variables \mathbf{q}_{rs} and, therefore, it has translational invariance, representing a homogeneous state. All of the time dependence occurs through the thermal velocity defined by Eqs. (5) and (3), so the Liouville equation becomes for $\rho_h^*(\Gamma^*)$

$$\bar{\mathcal{L}}^* \rho_h^*(\Gamma^*) = 0. \quad (17)$$

The operator $\bar{\mathcal{L}}^*$ is the sum of the original generator for trajectories, now in the dimensionless variables, plus a scaling operator representing the time dependenc of $v_0(t)$ using Eq. (3),

$$\bar{\mathcal{L}}^* = \bar{L}^* + \frac{\zeta_0^*}{2} \sum_{r=1}^N \frac{\partial}{\partial \mathbf{v}_r^*} \cdot \mathbf{v}_r^*, \quad (18)$$

where ζ_0^* is the dimensionless cooling rate

$$\zeta_0^* = \frac{\ell}{v_0(t)} \zeta_0 [T_h(t)]. \quad (19)$$

The solution to Eq. (17) will be referred to as the homogeneous cooling state (HCS).

For more general solutions to the Liouville equation, it is useful to introduce the same scaled velocities to account for this inherent cooling. Then, the Liouville equation in dimensionless form becomes

$$\left(\frac{\partial}{\partial s} + \bar{\mathcal{L}}^*\right) \rho^*(\Gamma^*, s) = 0. \quad (20)$$

In this form, it is seen that the HCS is a stationary solution to the dimensionless Liouville equation. This is an important result for the representation of granular response functions. It shows that, in the appropriate dimensionless form, the reference state is again stationary, just as for equilibrium fluids. However, the introduction of this stationary representation comes at the price of changing the generator for the dynamics from \bar{L}^* to $\bar{\mathcal{L}}^*$. For the purposes of the discussion here, it is assumed that all properties of this homogeneous reference state are known. Further comments on the HCS are given in the Appendix A.

The deviations of the relative hydrodynamic fields from their values in the HCS, $\delta\tilde{y}_\beta^*(\mathbf{k}^*, s)$, are the averages of associated phase functions $\tilde{a}_\beta^*(\Gamma^*, s)$,

$$\delta\tilde{y}_\beta^*(\mathbf{k}^*, s) = \int d\Gamma^* \tilde{a}_\beta^*(\Gamma^*; \mathbf{k}^*) [\rho^*(\Gamma^*, s) - \rho_h^*(\Gamma^*)], \quad (21)$$

with

$$\tilde{a}_\beta^*(\Gamma^*; \mathbf{k}^*) = \frac{1}{n_h^*} \sum_{r=1}^N e^{i\mathbf{k}^* \cdot \mathbf{q}_r^*} a_\beta^*(\mathbf{v}_r^*), \quad (22)$$

where $n_h^* \equiv n_h \ell^d$ is the number density of the HCS in the reduced units and the single particle functions $a_\beta^*(\mathbf{v}_r^*)$ are defined by

$$\{a_\beta^*(\mathbf{v})\} \equiv \left\{1, \frac{2v^{*2}}{d} - 1, v_\parallel^*, \mathbf{v}_\perp^*\right\}. \quad (23)$$

For later convenience, the components of the flow field and the velocity of the particles have been chosen to be a longitudinal component along \mathbf{k}^* , and $d - 1$ transverse components, so that $v_\parallel^* \equiv \hat{\mathbf{k}} \cdot \mathbf{v}^*$ and $v_{\perp i}^* \equiv \hat{\mathbf{e}}_i \cdot \mathbf{v}^*$, with $\{\hat{\mathbf{k}} \equiv \mathbf{k}^*/k^*, \hat{\mathbf{e}}_i; i = 1, \dots, d - 1\}$ forming a set of d pairwise perpendicular unit vectors. To formulate the linear response problem, the initial state $\rho^*(\Gamma^*, 0)$ for the solution to the Liouville equation in (20) is chosen to be close to the HCS, in the sense that it is a functional of the initial fields $\delta y_\beta^*(\mathbf{r}^*, 0)$ and becomes the HCS for $\delta y_\beta^* \rightarrow 0$. More specifically, the system is viewed as partitioned into small cells such that the distribution function is the HCS in each cell, but with different values for the hydrodynamic fields. This is the analogue of the local equilibrium distribution for normal fluids, and will be referred to as the *local* HCS, $\rho_{lh}^*[\Gamma^*|\delta y^*]$. Its construction is discussed in

Appendix A. For small δy_β^* , the solution to the Liouville equation for the initial local HCS expanded to first order is

$$\begin{aligned}\rho^*(\Gamma^*, s) - \rho_h^*(\Gamma^*) &= e^{-s\mathcal{L}^*} \rho^*(\Gamma^*, 0) \\ &\rightarrow e^{-s\mathcal{L}^*} \sum_{\gamma=1}^{d+2} \int d\mathbf{r}^* \left[\frac{\delta \rho_{\ell h}^* [\Gamma^* | \delta y^*]}{\delta y_\gamma^* (\mathbf{r}^*)} \right]_{\delta y^*=0} \delta y_\gamma^* (\mathbf{r}^*, 0).\end{aligned}\quad (24)$$

Substitution into Eq. (21) allows identification of the response functions $C_{\beta\gamma}^*(\mathbf{r}^*, s)$ and their equivalent Fourier representation (using translational invariance of the functional derivative at $\delta y^* = 0$),

$$\tilde{C}_{\beta\gamma}^*(\mathbf{k}, s) = \int d\Gamma^* \tilde{a}_\beta^*(\Gamma^*; \mathbf{k}^*) \psi_\gamma^*(\Gamma^*; \mathbf{0}^*, s), \quad (25)$$

with

$$\psi_\gamma^*(\Gamma^*; \mathbf{r}^*, s) = e^{-s\bar{\mathcal{L}}^*} \psi_\gamma^*(\Gamma^*; \mathbf{r}^*) \quad (26)$$

and

$$\psi_\gamma^*(\Gamma^*; \mathbf{r}^*) = \left[\frac{\delta \rho_{\ell h}^* [\Gamma^* | \delta y^*]}{\delta y_\gamma^* (\mathbf{r}^*)} \right]_{\delta y=0}. \quad (27)$$

These results are closely analogous to those for a normal fluid, where $\rho_{\ell h}^*[\Gamma^* | \delta y^*]$ is an initial local equilibrium ensemble and $\rho_h^*(\Gamma^*)$ is the reference equilibrium Gibbs ensemble, $\rho_e^*(\Gamma^*)$. For example, if the grand canonical ensemble were used, the $\tilde{\psi}_\gamma^*$'s would become linear combinations of the set $\{\tilde{a}_\gamma^*\}$ times the equilibrium ensemble, and the response functions would be equilibrium time correlation functions for the local conserved quantities.

III. KINETIC THEORY

All of the following analysis is carried out in terms of the dimensionless variables, so the asterisk will be left implicit for simplicity. The response functions of Eq. (25) are expressed in terms of the full N -particle phase space. A reduced description in terms of the single particle phase space is possible because the functions $\tilde{a}_\beta(\Gamma, \mathbf{k})$ defined in Eq. (22) are sums of single particle functions. Consequently, integrating over the positions and momenta for all except one particle leads to the exact alternative representation:

$$\tilde{C}_{\beta\gamma}(\mathbf{k}, s) = \frac{1}{n_h} \int d\mathbf{v}_1 a_\beta(\mathbf{v}_1) \tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{k}, s), \quad (28)$$

where

$$\tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{k}, s) = \int d\mathbf{q}_1 e^{i\mathbf{k} \cdot \mathbf{q}_1} \psi_\gamma^{(1)}(x_1, s). \quad (29)$$

The function $\psi_\gamma^{(1)}(x_1, s)$ is the first element of a hierarchy of N functions $\psi_\gamma^{(m)}(x_1, x_2, \dots, x_m, s)$, $m = 1, \dots, N$, defined through

$$\psi_\gamma^{(m)}(x_1, \dots, x_m, s) \equiv \left[\frac{\delta f^{(m)}(x_1, \dots, x_m, s)}{\delta y_\gamma(\mathbf{0}, 0)} \right]_{\delta y=0}, \quad (30)$$

$$f^{(m)}(x_1, \dots, x_m, s) \equiv \frac{N!}{(N-m)!} \int dx_{m+1} \dots dx_N \rho(\Gamma, s), \quad (31)$$

$1 \leq m \leq N$. The functions $f^{(m)}(x_1, \dots, x_m, s)$ are the reduced distribution functions associated with the solution to the Liouville equation (20). They obey the corresponding BBGKY hierarchy of equations [4]. The first equation of this hierarchy is

$$\left(\frac{\partial}{\partial s} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} \right) f^{(1)}(x_1, s) + \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 f^{(1)}(x_1, s)] = \int dx_2 \bar{T}(x_1, x_2) f^{(2)}(x_1, x_2, s). \quad (32)$$

Then, it follows directly from the definition (30) that the function $\psi_\gamma^{(1)}(x_1, s)$ obeys the analogous equation

$$\left(\frac{\partial}{\partial s} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} \right) \psi_\gamma^{(1)}(x_1, s) + \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 \psi_\gamma^{(1)}(x_1, s)] = \int dx_2 \bar{T}(x_1, x_2) \psi_\gamma^{(2)}(x_1, x_2, s). \quad (33)$$

The representation given by Eq. (28) is very appealing, since the N -particle problem has been expressed without approximation in terms of the effective dynamics in the single particle phase space. The fundamental difficulty, however, is that the first hierarchy equation (33) does not determine this effective dynamics without specifying $\psi_\gamma^{(2)}(x_1, x_2, s)$. An equation similar to (33) can be written for $\psi_\gamma^{(2)}(x_1, x_2, s)$, the second BBGKY hierarchy equation, but it in turn requires specification of $\psi_\gamma^{(3)}(x_1, x_2, x_3, s)$. In this way, a coupling to the full N particle problem recurs. This coupling is broken if $\psi_\gamma^{(2)}(x_1, x_2, s)$ can be specified as an explicit functional of $\psi_\gamma^{(1)}(x_1, s)$. It is argued in Appendix B that this functional, when it exists, is independent of $\delta y_\gamma(\mathbf{r}, 0)$, linear in $\psi_\gamma^{(1)}(x, s)$, and has the general form

$$\psi_\gamma^{(2)}(x_1, x_2, s) = \int dx K(x_1, x_2, s; x) \psi_\gamma^{(1)}(x, s). \quad (34)$$

The kernel defining the functional is

$$K(x_1, x_2, s; x) = \left[\frac{\delta f^{(2)}(x_1, x_2, s)}{\delta f^{(1)}(x, s)} \right]_{\delta y=0}. \quad (35)$$

Once $K(x_1, x_2, s; x)$ is known, Eq. (33) becomes a closed, deterministic, linear kinetic equation for $\psi_\gamma^{(1)}$ in its most general form,

$$\left[\frac{\partial}{\partial s} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} + M(s) \right] \psi_\gamma^{(1)}(x_1, s) + \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 \psi_\gamma^{(1)}(x_1, s)] = 0, \quad (36)$$

with the formal “collision operator” M given by

$$M(x_1, s) \psi_\gamma^{(1)}(x_1, s) \equiv - \int dx \left[\int dx_2 \bar{T}(x_1, x_2) K(x_1, x_2, s; x) \right] \psi_\gamma^{(1)}(x, s). \quad (37)$$

This notion of a kinetic equation makes no *a priori* assumptions regarding the density or degree of dissipation, as is sometimes assumed. In fact, it is formally exact so any restrictions arise only when specific approximations are introduced to construct the functional.

The utility of the kinetic theory representation in any specific application depends on an accurate construction of the kernel $K(x_1, x_2, s; x)$ defined in Eq. (35). This is the point at which the full many-body problem must be confronted. The idea originated with Bogoliubov [14], who conceived that all $f^{(m)}(x_1, \dots, x_m, s)$ with $m > 1$ take a simpler form, $f^{(m)}[x_1, \dots, x_m | f^{(1)}(s)]$, after a brief “synchronization” time, where all time dependence occurs only through $f^{(1)}(s)$. For normal fluids at low density, construction of this functional can be accomplished by formal density expansions, leading to a sequence of contributions from clusters of particles of increasing size [15]. At lowest order, the closure is that associated with the Boltzmann equation, while at next order three particle scattering is described. Even for normal fluids, these sophisticated cluster expansions have limited direct use due to many-particle recollisions events that contribute secular terms in the formal expansions, violating the notion of a short synchronization time [16]. These recollisions (“rings”) in turn signal non-analytic density dependence and slow algebraic decay of correlations in time [17]. The experience gained in such studies over the past forty years has provided important insight for the construction of more phenomenological closures, such as the Enskog approximation for moderately dense gases and mode coupling models for very dense and metastable (glassy) fluids [18]. The development of a kinetic theory for granular fluids provides an opportunity to revisit many of these issues in an even more challenging context [19]. The analysis of the next sections illustrates this for the simplest “mean field” approximation to describe density effects beyond the Boltzmann limit.

IV. THE MARKOVIAN APPROXIMATION

The approximation described in this section is based on the assumption that the form of the correlations and their effect on collisional properties are essentially the same at all times. If so, the kernel $K(x_1, x_2, s; x)$ that determines these properties at the two-particle

level, can be represented approximately by its form at the initial time $s = 0$,

$$K(x_1, x_2, s; x) \simeq K(x_1, x_2, 0; x). \quad (38)$$

The collision operator term in Eq. (37) then becomes

$$M(x_1)\psi_\gamma^{(1)}(x_1, s) = - \int dx \int dx_2 \bar{T}(x_1, x_2) \left[\frac{\delta f_{\ell h}^{(2)}[x_1, x_2|\delta y]}{\delta f_{\ell h}^{(1)}[x|\delta y]} \right]_{\delta y=0} \psi_\gamma^{(1)}(x, s), \quad (39)$$

where it has been used that the initial conditions $f^{(m)}(x_1, \dots, x_m, 0)$ are the corresponding reduced distribution functions associated with the local HCS, $\rho_{\ell h}[\Gamma|\delta y]$, given in Eq. (A8),

$$f^{(m)}(x_1, \dots, x_m, 0) = f_{\ell h}^{(m)}[x_1, \dots, x_m|\delta y] = \frac{N!}{(N-m)!} \int dx_{m+1} \dots dx_N \rho_{\ell h}[\Gamma|\delta y]. \quad (40)$$

Since the approximate operator $M(x_1)$ is time-independent, the entire generator for the dynamics of $\psi_\gamma^{(1)}(x_1, s)$ in Eq. (36) also is independent of time. The equation is written in compact form as

$$\left[\frac{\partial}{\partial s} + \Lambda(x_1) \right] \psi_\gamma^{(1)}(x_1, s) = 0, \quad (41)$$

with the generator $\Lambda(x_1)$ given by

$$\Lambda(x_1)X(x_1) \equiv \left[\mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} + M(x_1) \right] X(x_1) + \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 X(x_1)], \quad (42)$$

for arbitrary $X(x_1)$. This is a necessary condition for a Markovian description and, consequently, Eq. (38) will be referred to as the Markovian approximation. The resulting kinetic theory is exact at asymptotically short times, and the nature of the approximation makes no explicit limitation on the density or the degree of dissipation. Of course, two particle correlations that develop over time are neglected. For a normal fluid, this Markov approximation leads to the Enskog approximation, where only time independent two-particle correlations are taken into account. The neglected time-dependent correlations are found in that case to be important only at high densities, and the Enskog approximation provides relevant corrections to the Boltzmann results up to moderate densities. It is reasonable to expect a similar context for granular fluids, although conditioned by the additional parameter space of the coefficient of restitution α .

The construction of $K(x_1, x_2, 0; x)$ now involves only analysis of the initial reduced distribution functions associated with the local HCS, and the corresponding collision operator

M is entirely characterized by properties of this state. It is useful to make this more explicit in terms of the pair correlation function $g_{\ell h}^{(2)}[x_1, x_2|\delta y]$ for the HCS, defined by

$$f_{\ell h}^{(2)}[x_1, x_2|\delta y] \equiv f_{\ell h}^{(1)}[x_1|\delta y] f_{\ell h}^{(1)}[x_2|\delta y] g_{\ell h}^{(2)}[x_1, x_2|\delta y]. \quad (43)$$

Then Eq. (35) in the Markovian approximation yields

$$\begin{aligned} K(x_1, x_2, 0; x) = & g_h^{(2)}(x_1, x_2) \left[f_h^{(1)}(v_1) \delta(x - x_2) + f_h^{(1)}(v_2) \delta(x - x_1) \right] \\ & + f_h^{(1)}(v_1) f_h^{(1)}(v_2) \left[\frac{\delta g_{\ell h}^{(2)}[x_1, x_2|\delta y]}{\delta f_{\ell h}^{(1)}[x|\delta y]} \right]_{\delta y=0}. \end{aligned} \quad (44)$$

The first term on the right side is given explicitly in terms of the HCS correlations, while the second term requires further analysis of the dependence of local HCS correlations on $f_{\ell h}^{(1)}$. For a *normal fluid*, these correlations are independent of the velocities and depend on the local equilibrium distribution $f_{\ell e}^{(1)}$ only through the local density,

$$\left[\frac{\delta g_{\ell h}^{(2)}[x_1, x_2|\delta y]}{\delta f_{\ell h}^{(1)}[x|\delta y]} \right]_{\delta y=0} \rightarrow \frac{1}{n_h} \left[\frac{\delta g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta y]}{\delta n(\mathbf{q})} \right]_{\delta y=0}. \quad (45)$$

The functional form of the local equilibrium pair correlation functional $g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta y]$ is known, so explicit construction of the collision operator M is possible in this case. The result is the linear revised Enskog kinetic equation [20]. For granular fluids, $g_{\ell h}^{(2)}[x_1, x_2|\delta y]$ has a more general functional dependence on $f_{\ell h}^{(1)}$ through its explicit functional dependence on δy , since

$$\delta y_\gamma(\mathbf{q}) = \frac{1}{n_h} \int dx_1 a_\gamma(x_1) \delta(\mathbf{q} - \mathbf{q}_1) \left\{ f_{\ell h}^{(1)}[x_1|\delta y] - f_h^{(1)}(v_1) \right\}, \quad (46)$$

where the $a_\gamma(x_1)$'s are the single particle functions given in Eq. (23). Thus

$$\left[\frac{\delta g_{\ell h}^{(2)}[x_1, x_2|\delta y]}{\delta f_{\ell}^{(1)}[x|\delta y]} \right]_{\delta y=0} = \frac{1}{n_h} \sum_\lambda \left[\frac{\delta g_{\ell h}^{(2)}[x_1, x_2|\delta y]}{\delta y_\lambda(\mathbf{q})} \right]_{\delta y=0} a_\lambda(\mathbf{v}). \quad (47)$$

This provides the practical route for constructing the Markovian kinetic theory for granular fluids. The collision operator becomes

$$\begin{aligned} M(x_1) \psi_\gamma^{(1)}(x_1, s) = & - \int dx_2 \bar{T}(x_1, x_2) g_h^{(2)}(x_1, x_2) \left[f_h^{(1)}(v_1) \psi_\gamma^{(1)}(x_2, s) + f_h^{(1)}(v_2) \psi_\gamma^{(1)}(x_1, s) \right] \\ & - \sum_\lambda \int d\mathbf{q}_2 c_\lambda(\mathbf{v}_1, \mathbf{q}_{12}) \frac{1}{n_h} \int d\mathbf{v}_2 a_\lambda(\mathbf{v}_2) \psi_\gamma^{(1)}(x_2, s), \end{aligned} \quad (48)$$

with

$$c_\lambda(\mathbf{v}_1, \mathbf{q}_{12}) = \int dx \bar{T}(x_1, x) f_h^{(1)}(v_1) f_h^{(1)}(v) \left[\frac{\delta g_{\ell h}^{(2)}[x_1, x | \delta y]}{\delta y_\lambda(\mathbf{q}_2)} \right]_{\delta y=0}. \quad (49)$$

Further discussion and simplification of $c_\lambda(\mathbf{v}_1, \mathbf{q}_{12})$ is given in Appendix C.

The response functions of Eq. (28) are given in terms of $\tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, -\mathbf{k}, s)$, that is proportional to the Fourier transform of $\psi_\gamma^{(1)}(x_1, s)$ as indicated in Eq. (29). An equation for the latter can be easily derived from Eq. (36),

$$\left[\frac{\partial}{\partial s} + \tilde{\Lambda}(\mathbf{k}) \right] \tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, -\mathbf{k}, s) = 0, \quad (50)$$

where the generator for the dynamics is the Fourier transform of Λ defined in Eq. (42),

$$\tilde{\Lambda}(\mathbf{k}) = -i\mathbf{k} \cdot \mathbf{v}_1 + \widetilde{M}(\mathbf{k}) + \frac{\zeta_0}{2} \left(d + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} \right). \quad (51)$$

with $\widetilde{M}(\mathbf{v}_1, \mathbf{k})$ defined by

$$\begin{aligned} \widetilde{M}(\mathbf{v}_1, \mathbf{k}) X(\mathbf{v}_1) &\equiv - \int dx_2 \bar{T}(x_1, x_2) g_h^{(2)}(x_1, x_2) \left[e^{i\mathbf{k} \cdot \mathbf{q}_{12}} f_h^{(1)}(v_1) X(\mathbf{v}_2) + f_h^{(1)}(v_2) X(\mathbf{v}_1) \right] \\ &\quad - \sum_\lambda \tilde{c}_\lambda(\mathbf{v}_1, \mathbf{k}) \frac{1}{n_h} \int d\mathbf{v}_2 a_\lambda(\mathbf{v}_2) X(\mathbf{v}_2), \end{aligned} \quad (52)$$

for arbitrary $X(\mathbf{v}_1)$, $\tilde{c}_\lambda(\mathbf{v}_1, \mathbf{k})$ being the Fourier transform of $c_\lambda(\mathbf{v}_1, \mathbf{q}_{12})$.

The response functions in Eq. (28) become

$$\tilde{C}_{\beta\gamma}(\mathbf{k}, s) = \frac{1}{n_h} \int d\mathbf{v}_1 a_\beta(\mathbf{v}_1) e^{-s\tilde{\Lambda}(\mathbf{k})} \tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{k}), \quad (53)$$

$$\tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, -\mathbf{k}) = \int d\mathbf{q}_1 e^{i\mathbf{k} \cdot \mathbf{q}_1} \left[\frac{\delta f_{\ell h}^{(1)}[x_1 | \delta y]}{\delta \tilde{y}_\gamma(\mathbf{0})} \right]_{\delta \tilde{y}=0}. \quad (54)$$

Equation (53) is the primary practical result of our analysis here. It provides a realistic kinetic theory description of the most fundamental time dependent fluctuations in a granular fluid: those induced by perturbations of the hydrodynamic fields. To appreciate the scope and generality of this result, note that the Markovian approximation is exact at short times for all \mathbf{k} , densities, and degrees of restitution. In this short time limit, Eq. (53) yields

$$\lim_{s \rightarrow 0} \tilde{C}_{\beta\gamma}(\mathbf{k}, s) = \lim_{s \rightarrow 0} (e^{-sN(\mathbf{k})})_{\beta\gamma}, \quad (55)$$

with

$$N_{\beta\gamma}(\mathbf{k}) = \frac{1}{n_h} \int d\mathbf{v}_1 a_\beta(\mathbf{v}_1) \tilde{\Lambda}(\mathbf{k}) \tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{k}). \quad (56)$$

At longer times, the Markovian approximation is expected to continue to provide a good approximation across this parameter space, since the derivation does not explicitly require any limitations on \mathbf{k} , densities, or degrees of restitution. This expectation is borne out in the elastic limit, where the linear Enskog theory for equilibrium time correlation functions is recovered, as discussed in the next section. In that case, comparisons with molecular dynamics simulations (wavevector dependent transport [21]) and neutron scattering experiments [22], confirm the accuracy and practical utility of this kinetic theory over a wide range of wavevectors and densities. A determination of the corresponding domain of accuracy for granular fluids, awaits similar comparisons with simulation and experiments, but there is no simple reason to expect qualitative rather than quantitative differences from normal fluids.

In Sec. VI, it is shown that the operator $\tilde{\Lambda}(\mathbf{k})$ contains the hydrodynamic modes in its spectrum for small k . Thus the response functions in Eq. (53) provide a means to study the transition from short time dynamics to a presumed dominant hydrodynamics at long times for granular fluids, in a manner similar to that done for normal fluids [4]. In addition, since it is valid for all \mathbf{k} , the nature of hydrodynamics beyond the Navier-Stokes approximation can be studied.

An important application of general linear response methods, is the derivation of Helfand and Green-Kubo expressions for the transport coefficients [8, 9]. These are formally exact results given in terms of time correlation functions. The above analysis for the response functions applies to these as well, and their dynamics in the Markovian approximation is generated by the same operator $\tilde{\Lambda}(\mathbf{k})$. The evaluation of the Helfand and Green-Kubo expressions for the shear viscosity is illustrated in Sec. VII. The results provide a generalization of those from Enskog kinetic theory [11], to include pair velocity correlations. When such correlations are neglected, the results of ref. [11] are recovered in detail. This is verified for the other Navier-Stokes transport coefficients as well in Appendix E.

V. GRANULAR ENSKOG APPROXIMATION

The Markovian approximation discussed in the previous section, requires specification of $g_{\ell h}^{(2)}(x_1, x_2)$ and $c_\lambda(x_1, \mathbf{q}_{12})$, or equivalently $\delta g_{\ell h}^{(2)}[x_1, x|\delta y]/\delta y_\lambda(\mathbf{q}_2)$ for $\delta y = 0$. While these are well defined in terms of the local HCS distribution, little is known about their detailed forms as yet, except in the elastic limit where they are accurately determined from liquid

state theory of the pair correlation function. The important simplification in that case is the absence of velocity correlations. It is plausible to assume that such correlations remain weak for the granular fluid as well, and to make the approximation

$$g_{\ell h}^{(2)}[x_1, x_2|\delta y] \simeq g_{\ell h}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta y] = g_{\ell h}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta n]. \quad (57)$$

The last equality recognizes that the neglect of velocity correlations leads to a functional that is independent of δT and $\delta \mathbf{U}$ (for hard spheres or disks) and hence is a functional only of the density. Furthermore, $g_{\ell h}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|0] = g_h^{(2)}(q_{12})$ as a consequence of fluid symmetry. Then, Eq. (49) reduces to

$$c_{\lambda}^{(E)}(\mathbf{v}_1, \mathbf{q}_{12}) = \delta_{\lambda 1} c^{(E)}(\mathbf{v}_1, \mathbf{q}_{12}) = \delta_{\lambda 1} \int dx \bar{T}(x_1, x) f_h^{(1)}(v_1) f_h^{(1)}(v) \left[\frac{\delta g_{\ell h}^{(2)}(\mathbf{q}_1, \mathbf{q}|\delta n)}{\delta n(\mathbf{q}_2)} \right]_{\delta n=0}. \quad (58)$$

The collision operator (52) now simplifies to

$$\begin{aligned} \widetilde{M}^E X(\mathbf{v}_1) = & -g_h^{(2)}(\sigma) \int dx_2 \bar{T}(x_1, x_2) \left[e^{-i\mathbf{k} \cdot \mathbf{q}_{12}} f_h^{(1)}(v_1) X(\mathbf{v}_2) + f_h^{(1)}(v_2) X(\mathbf{v}_1) \right] \\ & - \frac{1}{n_h} \widetilde{c}^{(E)}(\mathbf{v}_1, \mathbf{k}) \int d\mathbf{v}_2 X(\mathbf{v}_2). \end{aligned} \quad (59)$$

It remains to give the explicit density dependence for $g_{\ell h}^{(2)}[\mathbf{r}_1, \mathbf{r}|\delta n]$ [23]. As a practical matter, it can be chosen to be the pair distribution for a nonuniform normal fluid for which a well-developed theory exists. In that case, $g_h^{(2)}(|\mathbf{r}_1 - \mathbf{r}|) = g_{eq}^{(2)}(|\mathbf{r}_1 - \mathbf{r}|)$, the radial distribution function for a uniform hard sphere fluid. Also, for this choice the functional derivative appearing in the expression of $\widetilde{c}^{(E)}(\mathbf{v}_1, \mathbf{k})$ can be evaluated for the first few terms of a \mathbf{k} expansion, as is required for evaluation of transport coefficients. The context of such a choice would be that the static spatial correlations of a hard sphere system are due to excluded volume effects, and these can be captured using the pair correlation function of a fluid of elastic hard spheres. The generator $\Lambda(\mathbf{k})$ obtained in this approximation from Eq. (59) gives the granular Enskog kinetic theory, and is the linearized version of the one studied in ref. [11].

VI. HYDRODYNAMIC MODES

An important feature of the response functions considered here is their relationship to hydrodynamic response. At small k and large s , these response functions should correspond

to those from the phenomenological hydrodynamic equations. For example, it is this relationship that allows the derivation of Helfand and Green-Kubo expressions for the transport coefficients. Any acceptable approximate kinetic theory for the response functions should preserve this relationship to hydrodynamics. More specifically, the hydrodynamic excitations should appear in the spectrum of the linear operator $\tilde{\Lambda}(\mathbf{k})$,

$$\tilde{\Lambda}(\mathbf{k}, \mathbf{v}) \phi^{(\beta)}(\mathbf{k}, \mathbf{v}) = \lambda^{(\beta)}(\mathbf{k}) \phi^{(\beta)}(\mathbf{k}, \mathbf{v}), \quad \beta = 1, \dots, d+2, \quad (60)$$

where the set $\{\lambda^{(\beta)}(\mathbf{k})\}$ are the eigenvalues of the $d+2$ linearized hydrodynamic equations. The above eigenfunctions and eigenvalues are determined in the limit $\mathbf{k} = \mathbf{0}$ in Appendix D with the results

$$\{\lambda^{(\beta)}(\mathbf{0})\} = \left\{ 0, \frac{\zeta_0}{2}, -\frac{\zeta_0}{2} \right\}, \quad (61)$$

$$\{\phi^{(\beta)}(\mathbf{v}, \mathbf{0})\} = \left\{ \tilde{\psi}_1^{(1)}(\mathbf{v}, \mathbf{0}) - 2 \left(\frac{\partial \ln \zeta_0}{\ln n_h} \right)_{T_h} \tilde{\psi}_2^{(1)}(\mathbf{v}, \mathbf{0}), \tilde{\psi}_2^{(1)}(\mathbf{v}, \mathbf{0}), \tilde{\psi}_3^{(1)}(\mathbf{v}, \mathbf{0}) \right\}. \quad (62)$$

The eigenvalue $-\zeta_0/2$ is d -fold degenerate, and the associated eigenfunctions are the components of the vector $\tilde{\psi}_3^{(1)}(\mathbf{v}, \mathbf{0}) \equiv -\partial f_h^{(1)}(\mathbf{v})/\partial \mathbf{v}$.

In the elastic limit, these eigenvalues are all zero, corresponding to the $d+2$ conservation laws, and the eigenfunctions become Maxwellians times linear combinations of the summational invariants $(1, v^2, \mathbf{v})$. For inelastic collisions, the nonzero eigenvalues describe response of the cooling temperature to linear perturbations and growth of a constant velocity perturbation relative to the characteristic cooling thermal velocity. In both cases, these are also the eigenvalues of the phenomenological linearized hydrodynamic equations in the long wavelength limit.

With the eigenfunctions and eigenvalues known at $\mathbf{k} = 0$, their values for finite but small k can be obtained by perturbation theory. In this way the Navier-Stokes transport coefficients can be determined directly from the coefficients up through order k^2 . This direct calculation of the spectrum for the generator of a linear kinetic theory has been described in detail recently for the granular Boltzmann equation [24], and its extension to the Markovian kinetic theory given here is straightforward. Instead, the remainder of this presentation addresses the calculation of the transport coefficients from an approximate evaluation of their Helfand and Green-Kubo representations that have been obtained from linear response.

VII. HELFAND AND GREEN-KUBO EXPRESSIONS

As noted above, the exact response functions defined in Sec. II must agree with those of the linearized phenomenological hydrodynamic equations in the long wavelength and long time limit. This relationship allows identification of the parameters of those phenomenological equations in terms of the response functions in this limit. The results of this analysis for granular fluids has been given recently, leading to expressions for the transport coefficients in terms of certain time correlation functions derived from the response functions [9]. These correlation functions can be evaluated approximately by the Markov kinetic theory, to obtain explicit results for all transport coefficients appearing in the Navier-Stokes hydrodynamic equations. Further, neglecting the velocity correlations in the Markov theory, allows the evaluation of these quantities in the granular Enskog theory, reproducing the results reported in [11]. In this section, only the shear viscosity is considered as an example, while all remaining transport coefficients are analyzed in Appendix E.

The exact Helfand and Green-Kubo expressions for the shear viscosity are (dimensionless units are still assumed)

$$\eta = \lim \Omega_H^\eta(s) = \Omega_H^\eta(0) + \lim \int_0^s ds' \Omega_G^\eta(s'), \quad (63)$$

respectively, where the symbol \lim denotes the hydrodynamic limit of $V \rightarrow \infty$, followed by $s \rightarrow \infty$. The correlation functions in the above equation are defined by

$$\Omega_H^\eta(s) = -\frac{V^{-1}}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int d\Gamma H_{ij}(\Gamma) e^{-s(\bar{\mathcal{L}} + \frac{\zeta_0}{2})} \mathcal{M}_{\eta,ij}(\Gamma), \quad (64)$$

$$\Omega_G^\eta(s) = \frac{\partial}{\partial s} \Omega_H^\eta(s) = -\frac{V^{-1}}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int d\Gamma H_{ij}(\Gamma) e^{-s(\bar{\mathcal{L}} + \frac{\zeta_0}{2})} \Upsilon_{\eta,ij}(\Gamma). \quad (65)$$

Here, $H_{ij}(\Gamma)$ is the volume integrated momentum flux,

$$H_{ij}(\Gamma) = \sum_{r=1}^N v_{r,i} v_{r,j} + \sum_{r=1}^N \sum_{s \neq r}^N H_{ij}^{(2)}(x_r, x_s), \quad (66)$$

$$H_{ij}^{(2)}(x_r, x_s) = \frac{(1 + \alpha)\sigma}{4} \delta(q_{rs} - \sigma) \Theta(-\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}) (\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{lm})^2 \hat{q}_{rs,i} \hat{q}_{rs,j}, \quad (67)$$

$\mathcal{M}_{\eta,ij}$ is the traceless tensor

$$\mathcal{M}_{\eta,ij} = -\frac{1}{2} \sum_{r=1}^N \left(q_{ri} \frac{\partial}{\partial v_{r,j}} + q_{r,j} \frac{\partial}{\partial v_{r,i}} - \frac{2}{d} \delta_{ij} \mathbf{q}_r \cdot \frac{\partial}{\partial \mathbf{v}_r} \right) \rho_h(\Gamma), \quad (68)$$

and $\Upsilon_{\eta,ij}(\Gamma)$ is the associated Green-Kubo conjugate flux,

$$\Upsilon_{\eta,ij}(\Gamma) = - \left(\bar{\mathcal{L}} + \frac{\zeta_0}{2} \right) \mathcal{M}_{\eta,ij}(\Gamma). \quad (69)$$

It is seen in Eqs. (64) and (65) that the generator of dynamics is $\bar{\mathcal{L}} + \frac{\zeta_0}{2}$. This reflects the fact that the $k = 0$ mode $\lambda = -\zeta_0/2$, of Eq. (61), has been subtracted out.

The time independent contribution $\Omega_H^\eta(0)$ in the Green-Kubo expression, can be evaluated exactly from the definitions (66) and (68) with the result:

$$\Omega_H^\eta(0) = \frac{V^{-1}}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int dx_1 h_{ij}(\mathbf{v}_1) \frac{1}{2} \left(q_{1,i} \frac{\partial}{\partial v_{1,j}} + q_{1,j} \frac{\partial}{\partial v_{1,i}} - \frac{2}{d} \delta_{ij} \mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{v}_1} \right) f_h^{(1)}(v_1), \quad (70)$$

where

$$h_{ij}(\mathbf{v}) = v_i v_j + \int dx_1 \int dx_2 H_{ij}^{(2)}(x_1, x_2) K(x_1, x_2, 0; x). \quad (71)$$

The first term in the above expression of h_{ij} gives no contribution to $\Omega_H^\eta(0)$, from fluid symmetry. The second term can be recognized as being proportional to the average collision frequency, ν_{av} , as determined by the loss part of the right hand side of the hard sphere BBGKY hierarchy (32) specialized for the HCS,

$$\Omega_H^\eta(0) = \frac{(1 + \alpha)\sigma^2}{4(d^2 + 2d)} \nu_{av} \quad (72)$$

$$\nu_{av} = 2\sigma^{d-1} \int d\hat{\boldsymbol{\sigma}} \int d\mathbf{v}_1 \int d\mathbf{v}_2 \Theta(-\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12}) f_h^{(2)}(\boldsymbol{\sigma}, \mathbf{v}_1, \mathbf{v}_2). \quad (73)$$

A. Evaluation in the Markov Approximation

A complete evaluation of the correlation function $\Omega_H^\eta(s)$ is possible using the Markov kinetic theory. As in Sec. III, the correlation function can be given a representation in terms of one and two particle functions

$$\begin{aligned} \Omega_H^\eta(s) = & -\frac{V^{-1}}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int dx_1 v_{1,i} v_{1,j} \mathcal{M}_{\eta,ij}^{(1)}(x_1, s) \\ & - \frac{V^{-1}}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int dx_1 \int dx_2 H_{ij}^{(2)}(x_1, x_2) \mathcal{M}_{\eta,ij}^{(2)}(x_1, x_2, s), \end{aligned} \quad (74)$$

with the reduced functions $\mathcal{M}_{\eta,ij}^{(m)}(x_1, \dots, x_m, s)$ defined by

$$\mathcal{M}_{\eta,ij}^{(m)}(x_1, \dots, x_m, s) \equiv \frac{N!}{(N-m)!} \int dx_{m+1} \dots \int dx_N e^{-s(\bar{\mathcal{L}} + \frac{\zeta_0}{2})} \mathcal{M}_{\eta,ij}(\Gamma). \quad (75)$$

Similarly to the functions ψ_γ considered in Sec. III, the above functions obey a BBGKY hierarchy, the first equation of which is

$$\begin{aligned} \left(\frac{\partial}{\partial s} + \frac{\zeta_0}{2} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} \right) \mathcal{M}_{\eta,ij}^{(1)}(x_1, s) + \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot \left[\mathbf{v}_1 \mathcal{M}_{\eta,ij}^{(1)}(x_1, s) \right] \\ = \int dx_2 \bar{T}(x_1, x_2) \mathcal{M}_{\eta,ij}^{(2)}(x_1, x_2, s). \end{aligned} \quad (76)$$

The Markovian approximation in the present case is the same as that defined by Eqs. (34) and (38)

$$\mathcal{M}_{\eta,ij}^{(2)}(x_1, x_2, s) \simeq \int dx K(x_1, x_2, 0; x) \mathcal{M}_{\eta,ij}^{(1)}(x, s). \quad (77)$$

Then, Eq. (76) becomes the Markovian kinetic equation

$$\left(\frac{\partial}{\partial s} + \frac{\zeta_0}{2} + \Lambda \right) \mathcal{M}_{\eta,ij}^{(1)} = 0, \quad (78)$$

where the linear operator Λ is the same as defined in Eq. (42).

In this approximation, the Helfand expression of the shear viscosity of a hard sphere or disk granular fluid becomes

$$\begin{aligned} \eta = \lim \Omega_H^\eta(s) \simeq \frac{V^{-1}}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int dx_1 h_{ij}(\mathbf{v}_1) e^{-s(\Lambda + \frac{\zeta_0}{2})} \\ \times \frac{1}{2} \left(q_{1,i} \frac{\partial}{\partial v_{1,j}} + q_{1,j} \frac{\partial}{\partial v_{1,i}} - \frac{2}{d} \delta_{ij} \mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{v}_1} \right) f_h^{(1)}(v_1), \end{aligned} \quad (79)$$

where $h_{ij}(\mathbf{v})$ is defined in Eq. (71).

Next, the Green-Kubo expression for η in the Markov approximation can be identified from Eqs. (63) and (65),

$$\Omega_H^\eta(0) = \frac{V^{-1}}{d^2 + d + 2} \sum_{i=1}^d \sum_{j=1}^d \int dx_1 h_{ij}(\mathbf{v}_1) \frac{1}{2} \left(q_{1,i} \frac{\partial}{\partial v_{1,j}} + q_{1,j} \frac{\partial}{\partial v_{1,i}} - \frac{2}{d} \delta_{ij} \mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{v}_1} \right) f_h^{(1)}(v_1) \quad (80)$$

and

$$\Omega_G^\eta(s) = \frac{V^{-1}}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int dx_1 h_{ij}(\mathbf{v}_1) e^{-s(\Lambda + \frac{\zeta_0}{2})} \gamma_{ij}(\mathbf{v}_1). \quad (81)$$

The reduced conjugate flux γ_{ij} is

$$\gamma_{ij}(\mathbf{v}_1) = - \left(\Lambda + \frac{\zeta_0}{2} \right) \frac{1}{2} \left(q_{1,i} \frac{\partial}{\partial v_{1,j}} + q_{1,j} \frac{\partial}{\partial v_{1,i}} - \frac{2}{d} \delta_{ij} \mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{v}_1} \right) f_h^{(1)}(v_1). \quad (82)$$

Comparison of Eqs. (70) and (80) shows that $\Omega_H^\eta(0)$ is given exactly in the Markov approximation. The Green-Kubo representation for the shear viscosity requires the large s limit of

the integral over s in Eq. (63). It can be verified that $\Omega_G^\eta(s)$ has no invariant part, so that this limit is expected to exist. This issue is discussed in some detail in Appendix E. Then the Green-Kubo expression for shear viscosity can be written as

$$\eta = \Omega_H^\eta(0) + \frac{1}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int d\mathbf{v}_1 h_{ij}(\mathbf{v}_1) \mathcal{D}_{ij}(\mathbf{v}_1), \quad (83)$$

where $D_{ij}(\mathbf{v})$ is a solution to the integral equation

$$\left(\frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} + M + \frac{\zeta_0}{2} \right) \mathcal{D}_{ij}(\mathbf{v}) = \gamma_{ij}(\mathbf{v}). \quad (84)$$

Upon writing the above equation, it has been taken into account that terms involving spatial derivatives give a vanishing contribution to the expression of the shear viscosity. This is the traditional form in which expressions for transport coefficients are obtained from a Chapman Enskog expansion of a normal solution to the kinetic equation governing the dynamics of the system.

B. Evaluation in the Granular Enskog Approximation

The further neglect of velocity correlations in the collision operator M , leads to the granular Enskog approximation, i.e., the results given by Eqs. (79) and (81) apply with only the replacement Λ by Λ^E , with

$$\Lambda^E(x) \equiv \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{q}} + M^E(x) + \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v}, \quad (85)$$

and the operator M^E given by,

$$\begin{aligned} M^E X(x_1) &\equiv -g_h^2(\sigma) \int dx_2 \bar{T}(x_1, x_2) \left[f_h^{(1)}(v_1) X(x_2) + X(x_1) f_h^{(1)}(v_2) \right] \\ &\quad - \frac{1}{n_h} \int d\mathbf{q}_2 c_1(x_1, \mathbf{q}_2) \int d\mathbf{v}_2 X(x_2). \end{aligned} \quad (86)$$

The function c_1 is given in Eq. (58). In the Enskog approximation, the conjugate flux γ_{ij} in Eq. (82) becomes

$$\gamma_{ij}(\mathbf{v}_1) = - \left(\Lambda^E + \frac{\zeta_0}{2} \right) \frac{1}{2} \left(q_{1,i} \frac{\partial}{\partial v_{1,j}} + q_{1,j} \frac{\partial}{\partial v_{1,i}} - \frac{2}{d} \delta_{ij} \mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{v}_1} \right) f_h^{(1)}(v_1). \quad (87)$$

One final simplification occurs for the shear viscosity and some other transport coefficients. The mean field term in Eq. (86) vanishes when acting on $\gamma_{ij}(\mathbf{v})$ and, therefore,

$$\left(\Lambda^E + \frac{\zeta_0}{2} \right) \gamma_{ij}(\mathbf{v}) = \left(\mathcal{J}(x) + \frac{\zeta_0}{2} \right) \gamma_{ij}(\mathbf{v}), \quad (88)$$

where the operator $\mathcal{J}(x)$ has been introduced,

$$\mathcal{J}(x) \equiv \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} - g_h^{(2)}(\sigma) \mathcal{I}. \quad (89)$$

Here, \mathcal{I} is the linearized Boltzmann collision operator for inelastic hard spheres or disks,

$$\mathcal{I}X(x_1) \equiv \int dx_2 \bar{T}(x_1, x_2) \left[f_h^{(1)}(v_1) X(x_2) + X(x_1) f_h^{(1)}(v_2) \right]. \quad (90)$$

Therefore, the correlation function in Eq. (81) and the expression for the shear viscosity in Eq. (83) take the final forms

$$\begin{aligned} \Omega_G^{\eta E}(s) &= \frac{1}{d^2 + d - 2} \left[1 + \frac{(1 + \alpha) \sigma^d n_h \pi^{d/2} g_h^{(2)}(\sigma)}{4\Gamma\left(\frac{d+4}{2}\right)} \right] \\ &\times \sum_{i=1}^d \sum_{j=1}^d \int d\mathbf{v} v_i v_j \exp \left[-s \left(\mathcal{J} + \frac{\zeta_0}{2} \right) \right] \gamma_{ij}(\mathbf{v}) \end{aligned} \quad (91)$$

and

$$\eta = \Omega_H^{\eta E}(0) + \frac{1}{d^2 + d - 2} \sum_{i=1}^d \sum_{j=1}^d \int d\mathbf{v} h_{ij}(\mathbf{v}) \mathcal{D}_{ij}^E(\mathbf{v}), \quad (92)$$

respectively. The Enskog approximation for $\Omega_H^\eta(0)$ is

$$\Omega_H^{\eta E}(0) = \frac{\pi^{(d-1)/2} (1 + \alpha) \sigma^{d+1} g_h^{(2)}(\sigma)}{2(d^2 + 2d) \Gamma\left(\frac{d+1}{2}\right)} \int d\mathbf{v}_1 \int d\mathbf{v}_2 |\mathbf{v}_1 - \mathbf{v}_2| f_h^{(1)}(v_1) f_h^{(1)}(v_2) \quad (93)$$

and \mathcal{D}_{ij}^E is a solution to the integral equation

$$\left(\mathcal{J} + \frac{\zeta_0}{2} \right) \mathcal{D}_{ij}^E(\mathbf{v}) = \gamma_{ij}(\mathbf{v}), \quad (94)$$

The above result for the shear viscosity agrees in detail with that obtained in ref. [11], through a Chapman-Enskog procedure applied to the non-linear granular Enskog equation.

The results in Eqs. (81) and (91) for the Green-Kubo integrand are new. At the formally exact level, the integrand is given by the correlation between the flux and the conjugate flux. In detail, the contributions from $H_{ij}^{(2)}$ and the $\bar{T}(i, j)$ terms of $\bar{\mathcal{L}}$, appear to yield singularities at $t = 0$, signaling a possible nonanalytic dependence on t . This occurs even in the elastic limit, and is a peculiarity of hard particle dynamics. Consequently, previous theoretical and simulation studies have avoided this by studying the Helfand forms for transport coefficients. The approximate kinetic theory described here gives an explicit analytic estimate for $\Omega_G^\eta(s)$, whose integral yields a good estimate for the transport coefficients. This suggests that $\Omega_G^\eta(s)$ may have a dominant analytic part with a relatively small non-analytic correction.

VIII. DISCUSSION

Kinetic theory has been used extensively as a formal tool for approximate evaluation of response functions, in the study of hard spheres as a model for normal fluids. The objective of this work is to take a first step in the development and application of this tool in the analogous field of granular fluids. The advantage of developing kinetic theory in the context of linear response functions, lies in the fact that the resulting theories are inherently linear, and provide a more tractable setting to explore questions such as aging to hydrodynamics and short wavelength behavior of the exact hydrodynamic response. The two primary contributions here are: 1) the development of a practical kinetic theory for an important class of granular time correlation functions and, 2) the demonstration of its utility for the evaluation of Helfand and Green-Kubo expressions for Navier-Stokes order transport coefficients.

The linear kinetic theory is summarized by Eq. (50). Based on corresponding studies for the elastic limit of this equation, it is expected to have a wide domain of validity with respect to space and time scales, as well as densities. The nature of the approximation, short time functional relationship, does not explicitly entail questions of inelasticity so it is expected to apply as well for a finite range of inelasticity. It encompasses the granular Boltzmann equation, in the low density limit, and the familiar Enskog equation in the elastic limit. The focus here has been on hydrodynamic response, but the theory includes hydrodynamics beyond the Navier-Stokes approximation, and even describes very short wavelength non-hydrodynamic behavior, that can be more important at moderate and high densities for granular fluids. Finally, this kinetic theory applies beyond the set of hydrodynamic fields considered here. For any observable, $z(\Gamma; \mathbf{r})$, that can be written as a sum of single particle functions so that

$$\tilde{z}(\Gamma, \mathbf{k}) = \sum_{r=1}^N e^{i\mathbf{k} \cdot \mathbf{q}_r} z(\mathbf{v}_r), \quad (95)$$

and for the same initial perturbation as considered in Sec. II, the response functions are given in the appropriate units by

$$\tilde{C}_\gamma(\mathbf{k}; s) = \frac{1}{n_h} \int d\mathbf{v} z(\mathbf{v}) e^{-s\tilde{\Lambda}(\mathbf{k})} \tilde{\psi}_\gamma^{(1)}(\mathbf{v}, \mathbf{k}). \quad (96)$$

This opens the possibility to study a wide range of experimental probes and also fundamental questions such as the relationship between fluctuations and response.

The second contribution, evaluation of the formal representations for Navier-Stokes transport coefficients, begins the process of exploring the utility of such formal representations, as well as verifying their consistency with earlier Chapman-Enskog based studies. These are long wavelength properties of the kinetic theory, and therefore a more controlled context for its tests. For example, the Markovian approximation provides a practical context for the introduction of velocity correlations associated with the reference homogeneous state [25, 26]. Their effect on transport coefficients at strong dissipation is expected to be important but has not been quantified to date. The kinetic theory scheme developed here also provides the basis for formulating and assessing more complex theories, such as those describing mode coupling dynamical correlations, which are expected to dominate at very high densities. Finally, the verification of the agreement between the kinetic theory evaluation of the Helfand and Green-Kubo representations here and the earlier Chapman-Enskog method provides further support for the implicit assumptions of these complementary formal approaches.

An interesting new result, both for normal and granular fluids, is the expression of the Green-Kubo time-correlation function $\Omega_G(s)$. In the Enskog approximation, the corresponding function for the shear viscosity is given by Eq. (91). To interpret this result, the time dependence may be estimated from a leading order cumulant expansion,

$$\Omega_G^{\eta E}(s) \simeq \Omega_G^{\eta E}(0) e^{-s/\tau}, \quad (97)$$

$$\frac{1}{\tau} = \frac{\sum_{i,j}^d \int d\mathbf{v} h_{ij}(\mathbf{v}) \left(\mathcal{J} + \frac{\zeta_0}{2}\right) \gamma_{ij}(\mathbf{v})}{\sum_{i,j}^d \int d\mathbf{v} h_{ij} \gamma_{ij}(\mathbf{v})}. \quad (98)$$

The corresponding Helfand correlation functions, $\Omega_H(s)$, can be inferred directly from this,

$$\Omega_H^{\eta E}(s) \simeq \Omega_H^{\eta E}(0) + \Omega_G^{\eta E}(0) \tau (1 - e^{-s/\tau}), \quad (99)$$

and the shear viscosity can be identified as

$$\eta \simeq \Omega_H^{\eta E}(0) + \Omega_G^{\eta E}(0) \tau. \quad (100)$$

These results expose the qualitative nature of the time dependence in each case. The resulting shear viscosity in these approximations agrees with that obtained by a leading order solution to the integral equation (94) as an expansion in Sonine polynomials [11].

In conclusion, it is hoped that this work provides a starting point to explore systematic analytic approximations to the hydrodynamic response of a granular fluid, with the same attention to detail given in the context of normal fluids. These, together with numerical studies of exact results, provide a means to understand transport mechanisms in this system.

IX. ACKNOWLEDGEMENTS

The research of A.B. and J.D. was supported in part by the Department of Energy Grant (DE-FG03-98DP00218). A.B. also acknowledges a McGinty Dissertation Fellowship and a IFT Michael J Harris Fellowship from the University of Florida. The research of J.J.B. was partially supported by the Ministerio de Educación y Ciencia (Spain) through Grant No. BFM2005-01398.

APPENDIX A: LOCAL HOMOGENEOUS STATE

The *local* HCS ensemble chosen as the initial perturbation of the HCS, represents a system decomposed into spatial cells, each in a HCS with its own local temperature, density, and flow velocity. It is constructed formally as follows. First, the HCS ensemble is determined as the solution to the homogeneous, stationary Liouville equation (17) in dimensionless form,

$$\rho_h^*(\Gamma^*) = \rho_h^* \left(\left\{ \frac{q_{rs}}{\ell}, \frac{\mathbf{v}_r - \mathbf{U}_h}{v_0(T_h)}; r, s = 1, \dots, N \right\} \right), \quad (\text{A1})$$

where \mathbf{U}_h , T_h , and n_h (not shown explicitly) are the flow field, temperature, and particle number density characterizing the HCS. Next, a conservative external force is added to Eq. (12) keeping the same $\overline{\mathcal{L}}^*$ operator,

$$\left\{ \overline{\mathcal{L}}^* - \sum_{r=1}^N \left[\frac{\partial}{\partial \mathbf{q}_r^*} \phi_{ext}^*(\mathbf{q}_r) \right] \cdot \frac{\partial}{\partial \mathbf{v}_r^*} \right\} \rho_h^{*'} = 0. \quad (\text{A2})$$

Here $\phi_{ext}^* \equiv \phi_{ext}/2T_h$, with $\phi_{ext}(\mathbf{r})$ being the potential associated with the external force. The solution of Eq. (A2) is, therefore, a function of this potential,

$$\rho_h^{*'} = \rho_h^{*'} \left(\left\{ \frac{q_{rs}}{\ell}, \frac{\mathbf{v}_r - \mathbf{U}_h}{v_0(t)}, \frac{\phi_{ext}(\mathbf{q}_r)}{T_h}; r, s = 1, \dots, N \right\} \right). \quad (\text{A3})$$

This can be considered as the nonuniform fluid ensemble corresponding to the uniform limit ρ_h^* , since in general the density will be nonuniform through its functional dependence on $\phi_{ext}(\mathbf{r})$,

$$n = n[\mathbf{r}|\phi_{ext}], \quad (\text{A4})$$

$$\phi_{ext} = \phi_{ext}[\mathbf{r}|n]. \quad (\text{A5})$$

The second equality assumes the functional dependence of the density on the external potential is invertible so that the potential can be expressed as a functional of the density field.

For normal fluids in the equilibrium Gibbs state, density functional theory assures that this is the case. In particular, for any chosen density field there is a unique external potential creating that field from the uniform state. It will be assumed that these properties hold as well here for the granular fluid, so that Eq. (A3) can be expressed in terms of the local density instead of the potential,

$$\rho_h^{*'} = \rho_h^{*''} \left(\left\{ \frac{q_{rs}}{\ell}, \frac{\mathbf{v}_r - \mathbf{U}_h}{v_0(T_h)}, n(\mathbf{q}_r) \ell^d; r, s = 1, \dots, N \right\} \right). \quad (\text{A6})$$

With $\rho_h^{*''}$ known from the solution to Eq. (A2), the local HCS is constructed by the replacements

$$\begin{aligned} \frac{\mathbf{v}_r - \mathbf{U}_h}{v_0(T_h)} &\rightarrow \frac{\mathbf{v}_r - \mathbf{U}_h - \delta \mathbf{U}(\mathbf{q}_r)}{v_0[T_h + \delta T(\mathbf{q}_r)]}, \\ n(\mathbf{q}_r) &\rightarrow n_h + \delta n(\mathbf{q}_r), \end{aligned} \quad (\text{A7})$$

to get

$$\rho_{\ell h}^* [\Gamma^* | \delta y^*] \equiv \rho_h^{*''} \left(\left\{ \frac{q_{rs}}{\ell}, \frac{\mathbf{v}_r - \mathbf{U}_h - \delta \mathbf{U}(\mathbf{q}_r)}{v_0 [T_h + \delta T(\mathbf{q}_r)]}, [n_h + \delta n(\mathbf{q}_r)] \ell^d; r, s = 1, \dots, N \right\} \right). \quad (\text{A8})$$

Note that the local HCS is no longer a solution to any Liouville equation, but rather is simply a reference ensemble representing an hypothetical HCS with different hydrodynamic parameters in each spatial cell of the fluid. Its construction in the way presented above, supports that interpretation in the sense that $\rho_{\ell h}^* [\Gamma^* | \delta y^* = \text{constant}] = \rho_h^* (\Gamma^*; y_h^* + \delta y^*)$ so that both $\{\rho_{\ell h}^* [\Gamma^* | \delta y^*]\}_{\delta y^*=0} = \rho_h^* (\Gamma^*; y_h^*)$ and all functional derivatives of $\rho_{\ell h}^* [\Gamma^* | \delta y^*]$ become derivatives of $\rho_h^* (\Gamma^*; y_h^*)$ at $\delta y^* = 0$. More explicitly, it is

$$\int d\mathbf{r}_1 \dots \int d\mathbf{r}_p \left[\frac{\delta^p \rho_{lh}[\Gamma | \delta y]}{\delta y_\alpha(\mathbf{r}_1) \dots \delta y_\beta(\mathbf{r}_p)} \right]_{y=\{n_h, T_h, \mathbf{0}\}} = \left[\frac{\partial^p \rho_h(\Gamma; n_h, T_h, \mathbf{U}_h)}{\partial y_{\alpha, h} \dots \partial y_{\beta, h}} \right]_{\mathbf{U}_h=\mathbf{0}}, \quad (\text{A9})$$

where Γ is a point in the phase space associated to the original positions and velocities and $\rho(\Gamma)$ the corresponding density. It is instructive to carry out this construction of the local ensemble for the case of a normal fluid. Then Eq. (A1) gives the familiar equilibrium Gibbs ensemble, and Eq. (A2) gives the same ensemble with the Hamiltonian modified to include the external potential. Finally, the construction in Eq. (A8) gives the familiar local equilibrium ensemble used in linear response theory for spatial perturbations of the equilibrium state. In the grand ensemble the dependence on the local density is implicit through a local chemical potential $\mu = \mu[\mathbf{r}|n]$.

APPENDIX B: TWO PARTICLE FUNCTIONAL

The reduced distribution functions associated with the solution to the Liouville equation (20) are defined as

$$f^{(m)}(x_1, \dots, x_m, s) \equiv \frac{N!}{(N-m)!} \int dx_{m+1} \dots \int dx_N e^{-s\bar{\mathcal{L}}} \rho_{\ell h}[\Gamma|\delta y]. \quad (\text{B1})$$

As is done in the main text, the asterisk indicating the use of dimensionless variables is left implicit. The $f^{(m)}$'s are clearly not independent functions. For example, $f^{(2)}(x_1, x_2, s)$ is related to $f^{(1)}(x_1, s)$ by

$$(N-1)f^{(1)}(x_1, s) = \int dx_2 f^{(2)}(x_1, x_2, s). \quad (\text{B2})$$

This implies that $f^{(2)}(x_1, x_2, s)$ has the representation

$$f^{(2)}(x_1, x_2, s) = f^{(1)}(x_1, s) f^{(1)}(x_2, s) g^{(2)}(x_1, x_2, s), \quad (\text{B3})$$

where the pair correlation function $g^{(2)}(x_1, x_2, s)$ has the properties

$$g^{(2)}(x_1, x_2, s) = g^{(2)}(x_2, x_1, s), \quad (\text{B4})$$

$$\int dx_2 f^{(1)}(x_2, s) g^{(2)}(x_1, x_2, s) = \int dx_1 f^{(1)}(x_1, s) g^{(2)}(x_1, x_2, s) = N-1. \quad (\text{B5})$$

This in turn shows that $g^{(2)}(x_1, x_2, s)$ is a functional of $f^{(1)}(x_1, s)$. Quite generally then, $f^{(2)}(x_1, x_2, s)$ can be considered a functional of $f^{(1)}(x_1, s)$,

$$f^{(2)}(x_1, x_2, s) = f^{(2)}[x_1, x_2, s | f^{(1)}(s)]. \quad (\text{B6})$$

However, this functional relationship is not unique. The utility of Eq. (B6) lies in discovering a choice that leaves the simplest functional dependence. In the low density limit, where it is expected that $g^{(2)}(x_1, x_2, s)$ tends to unity, this is clearly the case, since the functional becomes a constant independent of x_1, x_2 , and s . More generally, finding an appropriate functional form for $g^{(2)}(x_1, x_2, s)$ requires the detailed analysis of the full many-body problem.

The corresponding functional for $\psi_\gamma^{(2)}(x_1, x_2, s)$ defined by Eq. (30) can be computed as

$$\begin{aligned}
\psi_\gamma^{(2)}(x_1, x_2, s) &\equiv \left[\frac{\delta f^{(2)}(x_1, x_2, s)}{\delta y_\gamma(\mathbf{0}, 0)} \right]_{\delta y=0} \\
&= \int dx \left[\frac{\delta f^{(2)}[x_1, x_2, s | f^{(1)}(s)]}{\delta f^{(1)}(x, s)} \frac{\delta f^{(1)}(x, s)}{\delta y_\gamma(\mathbf{0}, 0)} \right]_{\delta y=0} \\
&= \int dx \left[\frac{\delta f^{(2)}[x_1, x_2, s | f^{(1)}(s)]}{\delta f^{(1)}(x, s)} \right]_{\delta y=0} \psi_\gamma^{(1)}(x, s) \\
&= \int dx K(x_1, x_2, s; x) \psi_\gamma^{(1)}(x, s), \tag{B7}
\end{aligned}$$

with $K(x_1, x_2, s; x)$ given by Eq. (35). This entails the additional requirement that the functional $f^{(2)}[x_1, x_2, s | \cdot]$ is independent of the specific initial fields $\delta y(\mathbf{r}, 0)$, i.e. $f^{(2)}[x_1, x_2, s | f^{(1)}(s)]$ depends on these fields only through $f^{(1)}(x, s)$. Consequently, $K(x_1, x_2, s; x)$ is also independent of such initial data. This is expected, since the collision operator $M(s)$ constructed from $K(x_1, x_2, s; x)$ by means of Eq. (37) should be universal for a wide class of initial conditions. Finally, the functional form for $\psi_\gamma^{(2)}(x_1, x_2, s)$ is seen to be linear in $\psi_\gamma^{(1)}(s)$, while in general $f^{(2)}[x_1, x_2, s | f^{(1)}(s)]$ is a nonlinear functional of $f^{(1)}(s)$.

APPENDIX C: INTERPRETATION OF $\tilde{c}_\lambda(\mathbf{v}, \mathbf{k})$

The contribution to the action of the collision operator $\widetilde{M}(\mathbf{k})$ on $\widetilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, -\mathbf{k}, s)$ from the term proportional to $\tilde{c}_\lambda(\mathbf{v}_1, \mathbf{k})$ in Eq. (52) is

$$-\sum_\lambda \tilde{c}_\lambda(\mathbf{v}_1, \mathbf{k}) \frac{1}{n_h} \int d\mathbf{v} a_\lambda(\mathbf{v}) \widetilde{\psi}_\gamma^{(1)}(\mathbf{v}, -\mathbf{k}, s) = -\sum_\lambda \tilde{c}_\lambda(\mathbf{v}_1, \mathbf{k}) \widetilde{C}_{\lambda\gamma}(\mathbf{k}; s), \tag{C1}$$

where Eq. (28) has been employed. The above contribution depends on only low order moments of the dependent variable $\widetilde{\psi}_\gamma^{(1)}$ in the kinetic equation (50), and in fact only those moments are of interest for determining the response functions. In this sense, \tilde{c}_λ is a mean field operator rather than a true collision operator, since its action does not depend directly on differences in $\widetilde{\psi}_\gamma^{(1)}$ before and after a collision like the first term of (52). Instead, Eq. (49) shows that \tilde{c}_λ reflects an average of collisional effects induced through changes in the correlations. To provide some interpretation of this term, consider first the elastic limit.

1. Elastic limit

In this case, $g_{\ell h}^{(2)}[x_1, x_2|\delta y] = g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta n]$, independent of the velocities, temperature, and flow field, i.e. it is a function of the spacial coordinates and a functional of the density. Equation (49) becomes

$$c_\lambda(\mathbf{v}_1, \mathbf{q}_{12}) = \delta_{\lambda 1} n_h^2 \int dx \bar{T}(x_1, x) \varphi(v_1) \varphi(v) \left[\frac{\delta g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}|\delta n]}{\delta n(\mathbf{q}_2)} \right]_{\delta n=0}, \quad (\text{C2})$$

where $\varphi(v)$ is the Maxwellian. The pair correlation function for a nonuniform fluid, $g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta n]$, appears in the stationary first BBGKY hierarchy equation (32) in the presence of an external potential ϕ_{ext} associated with the given density (see Appendix A),

$$\begin{aligned} & \left[\mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} - \left(\frac{\partial}{\partial \mathbf{q}_1} \phi_{ext}[\mathbf{q}_1|n] \right) \cdot \frac{\partial}{\partial \mathbf{v}_1} \right] n(\mathbf{q}_1) \varphi(v_1) \\ &= \int dx_2 \bar{T}(x_1, x_2) \varphi(v_1) \varphi(v_2) n(\mathbf{q}_1) n(\mathbf{q}_2) g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta n]. \end{aligned} \quad (\text{C3})$$

The functional derivative of this equation with respect to $\delta n(\mathbf{q}_3)$ evaluated at $\delta n = 0$ gives

$$\begin{aligned} n_h \varphi(v_1) \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} C(q_{13}) &= n_h \int dx_2 \bar{T}(x_1, x_2) \varphi(v_1) \varphi(v_2) g_e^{(2)}(q_{12}) [\delta(\mathbf{q}_{31}) + \delta(\mathbf{q}_{32})] \\ &+ n_h^2 \int dx_2 \bar{T}(x_1, x_2) \varphi(v_1) \varphi(v_2) \left[\frac{\delta g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta n]}{\delta n(\mathbf{q}_3)} \right]_{\delta n=0}. \end{aligned} \quad (\text{C4})$$

where it has been used that $g_e(q_{12}) = g_{\ell e}[\mathbf{q}_1, \mathbf{q}_2|0]$ and $c(q_{13})$ is the direct correlation function defined by [27]

$$n_h c(\mathbf{q}_1 - \mathbf{q}_3) = \delta(\mathbf{q}_1 - \mathbf{q}_3) + 2n_h \left[\frac{\delta \phi_{ext}[\mathbf{q}_1|n]}{\delta n(\mathbf{q}_3)} \right]_{\phi_{ext}=0}. \quad (\text{C5})$$

The first term on the right hand side of Eq. (C4) can be evaluated using the elastic limit of the explicit form for $\bar{T}(x_1, x_2)$ given in Eq. (13). The result is

$$\int dx_2 \bar{T}(x_1, x_2) \varphi(v_1) \varphi(v_2) g_e^{(2)}(q_{12}) [\delta(\mathbf{q}_{31}) + \delta(\mathbf{q}_{32})] = g_e^{(2)}(\sigma) \varphi(v_1) \delta(q_{13} - \sigma) \hat{\mathbf{q}}_{13} \cdot \mathbf{v}_1 \quad (\text{C6})$$

Finally, therefore, Eq. (C4) gives

$$n_h \int dx_2 \bar{T}(x_1, x_2) \varphi(v_1) \varphi(v_2) \left[\frac{\delta g_{\ell e}^{(2)}[\mathbf{q}_1, \mathbf{q}_2|\delta n]}{\delta n(\mathbf{q}_3)} \right]_{\delta n=0}$$

$$= \varphi(v_1) \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} [c(q_{13}) - g_e^{(2)}(\sigma) \Theta(q_{13} - \sigma)], \quad (\text{C7})$$

where the delta function in Eq. (C5) has been written in terms of the derivative of $\Theta(q_{13} - \sigma)$.

Using this into Eq. (C2) gives the desired result,

$$c_\lambda(\mathbf{v}_1, \mathbf{q}_{12}) = \delta_{\lambda 1} n_h \varphi(v_1) \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{q}_1} [c(q_{12}) - g_e^{(2)}(\sigma) \Theta(q_{12} - \sigma)]. \quad (\text{C8})$$

In the elastic case, it is seen that the contribution shown in Eq. (C1) is the same as that for an external force whose potential is $c(q_{13}) - g_h^{(2)}(\sigma) \Theta(q_{13} - \sigma)$. The direct correlation function has a discontinuity at $q_{13} = \sigma$, with value $c(\sigma) = g_e^{(2)}(\sigma)$, so the subtracted theta function contribution assures that this potential is continuous.

2. Inelastic collisions

For inelastic collisions, the effects of $\tilde{c}_\lambda(\mathbf{v}_1, \mathbf{k})$ are more complex and more difficult to interpret. However, a significant difference from the elastic case can be seen already for the simplest case of $\mathbf{k} = \mathbf{0}$. From Eq. (48), it follows that

$$\tilde{c}_\lambda(\mathbf{v}_1, \mathbf{0}) = \int d\mathbf{q}_{12} c_\lambda(x_1, \mathbf{q}_{12}) = \int dx \bar{T}(x_1, x) f_h^{(1)}(v_1) f_h^{(1)}(v) \left[\frac{\partial g_h^{(2)}(x_1, x; n_h, T_h, \mathbf{U}_h)}{\partial y_{\lambda, h}} \right]_{\mathbf{U}_h = \mathbf{0}}, \quad (\text{C9})$$

where use has been made of the identity

$$\int d\mathbf{q} \left[\frac{\delta g_{\ell h}^{(2)}[x_1, x_2 | \delta y]}{\delta y_\lambda(\mathbf{q})} \right]_{\delta y=0} = \left[\frac{\partial g_h^{(2)}(x_1, x; y)}{\partial y_\lambda} \right]_{\{n, T, \mathbf{U}\} = \{n_h, T_h, \mathbf{0}\}}. \quad (\text{C10})$$

This $\mathbf{k} = \mathbf{0}$ limit vanishes for elastic collisions, as it can be seen directly from Eq. (C8), but is nonzero for inelastic collisions. This can be verified in the Enskog approximation, where velocity correlations are neglected and, therefore,

$$\begin{aligned} \tilde{c}_\lambda^E(\mathbf{v}_1, \mathbf{0}) &= \delta_{\lambda 1} \left(\frac{\partial g_h^{(2)}(\sigma; n)}{\partial n} \right)_{n=n_h} \int dx \bar{T}(x_1, x) f_h^{(1)}(v_1) f_h^{(1)}(v) \\ &= \delta_{\lambda 1} \left(\frac{\partial \ln g_h^{(2)}(\sigma; n)}{\partial n} \right)_{n=n_h} \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 f_h^{(1)}(v_1)]. \end{aligned} \quad (\text{C11})$$

The second equality follows from the first hierarchy equation (32) particularized for the HCS,

$$\frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 f_h^{(1)}(v_1)] = \int dx \bar{T}(x_1, x) g_h^{(2)}(x_1, x) f_h^{(1)}(v_1) f_h^{(1)}(V), \quad (\text{C12})$$

when velocity correlations are neglected. Thus, it is seen that $\tilde{c}_\lambda(\mathbf{v}_1, \mathbf{0})$ includes changes in the correlations of collisional effects associated with cooling.

More generally, the Enskog approximation for arbitrary \mathbf{k} reads

$$\begin{aligned}\tilde{c}_\lambda^E(\mathbf{v}_1, \mathbf{k}) &= \delta_{\lambda 1} \int d\mathbf{q} e^{i\mathbf{k}\cdot\mathbf{q}} c_\lambda(x_1, \mathbf{q}) \\ &= \int dx_2 \bar{T}(x_1, x_2) f_h^{(1)}(v_1) f_h^{(1)}(v_2) \int d\mathbf{q} e^{i\mathbf{k}\cdot\mathbf{q}} \left[\frac{\delta g_{\ell h}^{(2)}[\mathbf{q}_1, \mathbf{q}_2 | \delta y]}{\delta n(\mathbf{q})} \right]_{\delta n=0}.\end{aligned}\quad (\text{C13})$$

APPENDIX D: HYDRODYNAMIC MODES OF $\tilde{\Lambda}(\mathbf{0})$

In this Appendix, some of the details leading to the solution of the eigenvalue problem (60) at $\mathbf{k} = \mathbf{0}$ are given. Consider first the functions $\tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{k})$, defined in Eq. (54), at $\mathbf{k} = \mathbf{0}$. By translational invariance

$$\left[\frac{\delta f_{\ell h}^{(1)}[\mathbf{q}_1, \mathbf{v}_1 | \delta y]}{\delta y_\gamma(\mathbf{0})} \right]_{\delta y=0} = \left[\frac{\delta f_{\ell h}^{(1)}[\mathbf{q}_1 + \mathbf{r}, \mathbf{v}_1 | \delta y]}{\delta y_\gamma(\mathbf{r})} \right]_{\delta y=0}, \quad (\text{D1})$$

and so

$$\begin{aligned}\tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{0}) &= \int d\mathbf{q}_1 \left[\frac{\delta f_{\ell h}^{(1)}[x_1 | \delta y]}{\delta y_\gamma(\mathbf{0})} \right]_{\delta y=0} = \frac{1}{V} \int d\mathbf{q}_1 \int d\mathbf{r} \left[\frac{\delta f_{\ell h}^{(1)}[x_1 | \delta y]}{\delta y_\gamma(\mathbf{r})} \right]_{\delta y=0} \\ &= \left[\frac{\partial f_h^{(1)}(\mathbf{v}; y)}{\partial y_\gamma} \right]_{y=y_h}.\end{aligned}\quad (\text{D2})$$

The last equality is a consequence of the construction of the local HCS, assuring that all functional derivatives in the homogeneous limit are related with ordinary derivatives of the HCS (see Appendix A). Use of the expression of the operator $\tilde{\Lambda}$, Eq. (51), yields

$$\tilde{\Lambda}(\mathbf{0}) \tilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{0}) = \tilde{M}(\mathbf{0}) \left[\frac{\partial f_h^{(1)}(\mathbf{v}_1; y)}{\partial y_\gamma} \right]_{y=y_h} + \frac{\zeta_0}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot \left\{ \mathbf{v}_1 \left[\frac{\partial f_h^{(1)}(\mathbf{v}_1; y)}{\partial y_\gamma} \right]_{y=y_h} \right\}. \quad (\text{D3})$$

Next, Eq. (52) gives

$$\begin{aligned}
\widetilde{M}(\mathbf{0}) \left[\frac{\partial f_h^{(1)}(\mathbf{v}_1; y)}{\partial y_\gamma} \right]_{y=y_h} &= - \int dx_2 \overline{T}(x_1, x_2) g_h^{(2)}(x_1, x_2) \left\{ \frac{\partial}{\partial y_\gamma} \left[f_h^{(1)}(\mathbf{v}_1; y) f_h^{(1)}(\mathbf{v}_2; y) \right] \right\}_{y=y_h} \\
&\quad - \frac{1}{n_h} \sum_\lambda \int dx \overline{T}(x_1, x) f_h^{(1)}(v_1) f_h^{(1)}(v) \left[\frac{\partial g_h^{(2)}(x_1, x; \delta y)}{\partial y_\lambda} \right]_{y=y_h} \\
&\quad \times \int d\mathbf{v}_2 a_\lambda(\mathbf{v}_2) \left[\frac{\partial f_h^{(1)}(\mathbf{v}_2; y)}{\partial y_\gamma} \right]_{y=y_h} \\
&= - \left[\frac{\partial}{\partial y_\gamma} \int dx_2 \overline{T}(x_1, x_2) g_h^{(2)}(x_1, x_2; y) f_h^{(1)}(\mathbf{v}_1; y) f_h^{(1)}(\mathbf{v}_2; y) \right]_{y=y_h}, \tag{D4}
\end{aligned}$$

where Eq. (C9) has been employed. The right hand side in the above equation can be further simplified by means of Eq. (C12), after changing \mathbf{v}_r into $\mathbf{v}_r - \mathbf{U}$,

$$\widetilde{M}(\mathbf{0}) \left[\frac{\partial f_h^{(1)}(\mathbf{v}_1; y)}{\partial y_\gamma} \right]_{y=y_h} = - \left\{ \frac{\partial}{\partial y_\gamma} \left[\frac{\zeta_0(y)}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot (\mathbf{v}_1 - \mathbf{U}) f_h^{(1)}(\mathbf{v}_1, y) \right] \right\}_{y=y_h}, \tag{D5}$$

and Eq. (D3) becomes

$$\widetilde{\Lambda}(\mathbf{0}) \widetilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{0}) = - \left[\frac{\partial \zeta_0(y)}{\partial y_\gamma} \right]_{y=y_h} \frac{1}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 f_h^{(1)}(v_1)] + \frac{\zeta_0}{2} \left(\frac{\partial \mathbf{U}}{\partial y_\gamma} \right)_{y=y_h} \cdot \frac{\partial}{\partial \mathbf{v}_1} f_h^{(1)}(v_1). \tag{D6}$$

Realizing that

$$\left[\frac{\partial f_h^{(1)}(\mathbf{v}, y)}{\partial y_2} \right]_{y=y_h} = - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} f_h^{(1)}(v)] \tag{D7}$$

and

$$\left[\frac{\partial f_h^{(1)}(\mathbf{v}, y)}{\partial \mathbf{y}_3} \right]_{y=y_h} = - \frac{\partial f_h^{(1)}(v)}{\partial \mathbf{v}}, \tag{D8}$$

Eq. (D6) is seen to be equivalent to

$$\widetilde{\Lambda}(\mathbf{0}) \widetilde{\psi}_\gamma^{(1)}(\mathbf{v}_1, \mathbf{0}) = \left[\frac{\partial \zeta_0(y)}{\partial y_\gamma} \right]_{y=y_h} \widetilde{\psi}_2^{(1)}(\mathbf{v}_1, \mathbf{0}) - \frac{\zeta_0}{2} \left(\frac{\partial \mathbf{U}}{\partial y_\gamma} \right)_{y=y_h} \cdot \widetilde{\psi}_3^{(1)}(\mathbf{v}_1, \mathbf{0}). \tag{D9}$$

Above, the index 3 and vectorial notation has been used to identify the d components associated to the velocity field.

The specific cases following from Eq. (D9) are

$$\widetilde{\Lambda}(\mathbf{0}) \widetilde{\psi}_1^{(1)}(\mathbf{v}_1, \mathbf{0}) = n_h \left(\frac{\partial \zeta_0}{\partial n_h} \right)_{T_h} \widetilde{\psi}_2^{(1)}(\mathbf{v}_1, \mathbf{0}), \tag{D10}$$

$$\tilde{\Lambda}(\mathbf{0})\tilde{\psi}_2^{(1)}(\mathbf{v}_1, \mathbf{0}) = \frac{\zeta_0}{2}\tilde{\psi}_2^{(1)}(\mathbf{v}_1, \mathbf{0}), \quad (\text{D11})$$

and

$$\tilde{\Lambda}(\mathbf{0})\tilde{\psi}_3^{(1)}(\mathbf{v}_1, \mathbf{0}) = -\frac{\zeta_0}{2}\tilde{\psi}_3^{(1)}(\mathbf{v}_1, \mathbf{0}). \quad (\text{D12})$$

Equations (D10) and (D11) can be combined to give

$$\tilde{\Lambda}(\mathbf{0}) \left[\tilde{\psi}_1^{(1)}(\mathbf{v}_1, \mathbf{0}) - 2 \left(\frac{\partial \ln \zeta_0}{\partial \ln n_h} \right)_{T_h} \tilde{\psi}_2^{(1)}(\mathbf{v}_1, \mathbf{0}) \right] = 0. \quad (\text{D13})$$

The above equations (D11)-(D13) have the form of eigenvalue equations

$$\tilde{\Lambda}(\mathbf{0})\phi^{(\beta)}(\mathbf{0}, \mathbf{v}_1) = \lambda^{(\beta)}(\mathbf{0})\phi^{(\beta)}(\mathbf{0}, \mathbf{v}_1), \quad (\text{D14})$$

$\beta = 1, \dots, d+2$, with the eigenvalues and eigenfunctions given in Eqs. (61) and (62).

APPENDIX E: TRANSPORT COEFFICIENTS

In this appendix, the pressure, the cooling rate, and the Navier-Stokes transport coefficients are evaluated in the kinetic theory approximations developed in the main text. These quantities are identified from the linearized phenomenological Navier-Stokes equations for an isolated granular fluid, that in dimensionless units have the form

$$\frac{\partial}{\partial s} \delta \tilde{y}_\beta^*(\mathbf{k}^*, s) + \sum_{\gamma=1}^{d+2} \mathcal{K}_{\beta\gamma}^{*hyd} \delta y_\gamma^*(\mathbf{k}^*, s) = 0, \quad (\text{E1})$$

where the transport matrix \mathcal{K}^{*hyd} is found to be block diagonal with a “longitudinal” part, corresponding to the fields $\{\delta n^*, \delta T^*, \delta U_\perp^*\}$, given by

$$\mathcal{K}^{*hyd}(\mathbf{k}^*) = \begin{pmatrix} 0 & 0 & -ik^* \\ \zeta_0^* \frac{\partial \ln \zeta_0}{\partial \ln n_h} + \left(\frac{2\mu^*}{d} - \zeta^{*n} \right) k^{*2} & \frac{\zeta_0^*}{2} + \left(\frac{2\lambda^*}{d} - \zeta^{*T} \right) k^{*2} & -i \left(\frac{2p_h^*}{d} + \zeta^{*U} \right) k^* \\ -\frac{ip_h^*}{2} \frac{\partial \ln p_h}{\partial \ln n_h} k^* & -\frac{ip_h^*}{2} k^* & -\frac{\zeta_0^*}{2} + \left[\frac{2(d-1)}{d} \eta^* + \kappa^* \right] k^{*2} \end{pmatrix}. \quad (\text{E2})$$

The “transverse” components, $\delta U_{\perp,i}^*$, decouple from the longitudinal ones, and their transport matrix reads

$$\mathcal{K}^{hyd*}(\mathbf{k}) = \left(-\frac{\zeta_0^*}{2} + \eta^* k^{*2} \right) I, \quad (\text{E3})$$

with I being the unit matrix of dimension $d-1$. These expressions include the unspecified functions $p_h(n_h, T_h)$ and $\zeta_h(n_h, T_h)$, defining the pressure and the cooling rate of the HCS,

respectively, as well as the unknown shear viscosity $\eta(n_h, T_h)$, the bulk viscosity $\kappa(n_h, T_h)$, the thermal conductivity $\lambda(n_h, T_h)$, and the new coefficient, $\mu(n_h, T_h)$, associated with the contribution of the density gradient to the heat transport in a granular fluid. Finally, $\zeta^n(n_h, T_h)$, $\zeta^T(n_h, T_h)$, and $\zeta^U(n_h, T_h)$ are transport coefficients arising from the local cooling rate. The dimensionless form of the cooling rate was defined in Eq. (19), and the definitions for the remaining dimensionless quantities are

$$\begin{aligned} p_h^* &= \frac{p_h}{n_h T_h}, \quad \eta^* = \frac{\eta}{m n_h l v_0}, \quad \kappa^* = \frac{\kappa}{m n_h l v_0}, \\ \lambda^* &= \frac{\lambda}{l n_h v_0}, \quad \mu^* = \frac{\mu}{l T_h v_0}, \\ \zeta^{*U} &= \zeta^U, \quad \zeta^{*n} = \frac{n_h \zeta^n}{l v_0}, \quad \zeta^{*T} = \frac{T_h \zeta^T}{l v_0}. \end{aligned} \quad (\text{E4})$$

Formal expressions for these parameters have been obtained in ref. [9], by carrying out a linear response analysis. Each of those expressions is evaluated below by means of the kinetic theory developed in the text, for comparison to the results obtained by a Chapman-Enskog solution to the nonlinear Enskog equation [11]. First, some general considerations that apply to all of the transport coefficients will be addressed. Dimensionless units will be used in the remaining of this Appendix, and the asterisk will be suppressed for simplicity, as done in the main text.

1. Reduced Form of the Transport Coefficients

The Helfand form for a generic dimensionless transport coefficient χ as obtained in ref. [9] is

$$\chi = \lim \Omega_H(s) = \lim \frac{1}{V} \int d\Gamma j(\Gamma) (1 - \mathcal{P}) e^{-s(\bar{\mathcal{L}} - \lambda)} \mathcal{M}(\Gamma), \quad (\text{E5})$$

where $j(\Gamma)$ is a flux associated with the densities of the hydrodynamic variables, and λ denotes one of the eigenvalues in (61). These fluxes have the generic form

$$j(\Gamma) = \sum_{r=1}^N j_1(x_l) + \sum_{r=1}^N \sum_{s \neq r}^N j_2(x_r, x_s), \quad (\text{E6})$$

j_1 and j_2 being one and two-particle functions of the phase point, respectively. The adjoint functions $\mathcal{M}(\Gamma)$ are related to the conjugate densities in the hydrodynamic response defined in Eq. (27), and read

$$\mathcal{M}(\Gamma) = M \int d\mathbf{r} \hat{\mathbf{k}} \cdot \mathbf{r} \left[\frac{\delta \rho_{lh}[\Gamma|y]}{\delta y(\mathbf{r})} \right]_{y=y_h}, \quad (\text{E7})$$

M being some constant. Moreover, \mathcal{P} is the projection operator defined by

$$\mathcal{P}(\Gamma)X(\Gamma) = \frac{1}{V} \sum_{\gamma=1}^{d+2} \tilde{\psi}_{\gamma}(\Gamma; \mathbf{0}) \int d\Gamma' \tilde{a}_{\gamma}(\Gamma'; \mathbf{0}) X(\Gamma'), \quad (\text{E8})$$

where the phase functions $\tilde{a}_{\gamma}(\Gamma; \mathbf{0})$ and $\tilde{\psi}_{\gamma}(\Gamma; \mathbf{0})$ are the densities defined by Eqs. (22) and the Fourier transform of (27), respectively, both evaluated at $\mathbf{k} = \mathbf{0}$. The generator for the dynamics is $\bar{\mathcal{L}} - \lambda$, where λ is one of the hydrodynamic modes at $\mathbf{k} = \mathbf{0}$ identified in Eq. (61). Both the projection operator and the additional time dependence of the term containing λ are necessary to ensure that the long time limit of the correlation function in Eq. (E5) is well defined.

Proceeding as in Sec. VII for the shear viscosity, a reduced expression for these transport coefficients in the Markov approximation can be obtained. The generic correlation function $\Omega_H(s)$ of Eq. (E5) becomes

$$\Omega_H(s) \simeq \frac{1}{V} \int dx_1 J(x_1) [1 - \mathcal{P}^{(1)}(x_1)] e^{-s(\Lambda - \lambda)} \mathcal{M}^{(1)}(x_1). \quad (\text{E9})$$

The direct flux $J(x_1)$ in this reduced time correlation function is

$$J(x) \equiv j_1(x) + \int dx_1 \int dx_2 j_2(x_1, x_2) K(x_1, x_2, 0; x), \quad (\text{E10})$$

where $K(x_1, x_2, 0; x)$ is the kernel for the collision operator given by Eq. (44), and $\mathcal{M}^{(1)}$ is the one-particle function in the hierarchy associated with \mathcal{M}

$$\mathcal{M}^{(1)}(x_1) = N \int dx_2 \dots \int dx_N \mathcal{M}(\Gamma). \quad (\text{E11})$$

The generator for the Markov dynamics Λ is given in (42). Lastly, $\mathcal{P}^{(1)}$ is the one particle analog of \mathcal{P} defined in Eq. (E8),

$$\mathcal{P}^{(1)}(x_1)X(x_1) \equiv \frac{1}{V} \sum_{\gamma=1}^{d+2} \tilde{\psi}_{\gamma}^{(1)}(\mathbf{v}_1, \mathbf{0}) \frac{1}{n_h} \int d\mathbf{x} a_{\gamma}(\mathbf{v}) X(\mathbf{x}), \quad (\text{E12})$$

where the $\tilde{\psi}_{\gamma}^{(1)}(\mathbf{v}_1, \mathbf{0})$ are the functions defined in Eq. (D2).

Equation (E5) can be transformed into the Green-Kubo form to give

$$\chi = \Omega_H(0) + \lim \int_0^s ds' \frac{\partial}{\partial s'} \Omega_H(s') \equiv \Omega_H(0) + \lim \int_0^s ds' \Omega_G(s'). \quad (\text{E13})$$

The Markov approximation in this representation follows directly from Eq. (E9),

$$\Omega_H(0) \simeq \frac{1}{V} \int dx_1 J(x_1) [1 - \mathcal{P}^{(1)}(x_1)] \mathcal{M}^{(1)}(x_1), \quad (\text{E14})$$

$$\Omega_G(s) \simeq \frac{1}{V} \int dx_1 J(x_1) [1 - \mathcal{P}^{(1)}(x_1)] e^{-s(\Lambda-\lambda)} \gamma(x_1), \quad (\text{E15})$$

with the conjugate flux $\gamma(x_1)$ given by

$$\gamma(x_1) \equiv -(\Lambda - \lambda) \mathcal{M}^{(1)}(x_1). \quad (\text{E16})$$

In this Green-Kubo form, the role of the projection operator can be readily interpreted as follows. Taking into account that $\mathcal{P}^{(1)}$ projects over the subspace spanned by the hydrodynamic eigenfunctions of $\tilde{\Lambda}(\mathbf{0})$, the property

$$[1 - \mathcal{P}^{(1)}(x_1)] e^{-s(\Lambda-\lambda)} = [1 - \mathcal{P}^{(1)}(x_1)] e^{-s(\Lambda-\lambda)} [1 - \mathcal{P}^{(1)}(x_1)], \quad (\text{E17})$$

is obtained. This shows that the presence of the projection operator in Eq. (E15) ensures that the generator of dynamics $\exp[-s(\Lambda - \lambda)]$ acts on a function that is orthogonal to its invariants and, therefore, has a well defined long time limit.

The above expressions can be specialized to the granular Enskog approximation by replacing the Markovian kernel K in Eq. (E10) with its Enskog approximation, and by replacing the linear operator Λ by Λ^E , defined in Eq. (85). Further, the reduced conjugate functions in (E11) and Eqs. (E12) in the Enskog approximation become

$$\{\mathcal{M}_\gamma^{(1)}(x)\} \simeq \left\{ M\hat{\mathbf{k}} \cdot \mathbf{q} f_h^{(1)}(v), \frac{M}{2} \hat{\mathbf{k}} \cdot \mathbf{q} \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} f_h^{(1)}(v)], -M\hat{\mathbf{k}} \cdot \mathbf{q} \frac{\partial}{\partial \mathbf{v}} f_h^{(1)}(v) \right\} \quad (\text{E18})$$

and

$$\{\psi_\gamma^{(1)}(\mathbf{v}, \mathbf{0})\} \simeq \left\{ f_h^{(1)}(v), \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} f_h^{(1)}(v)], -\frac{\partial}{\partial \mathbf{v}} f_h^{(1)}(v) \right\}, \quad (\text{E19})$$

respectively. Now, $f_h^{(1)}(v)$ is the one-particle distribution of the HCS obtained from the nonlinear Enskog kinetic equation. In the rest of the appendix, explicit expressions for the hydrodynamic parameters are given in the granular Enskog approximation.

2. Evaluation in the Enskog Approximation

The Helfand and Green-Kubo expressions for the Navier-Stokes transport coefficients of a hard sphere granular fluid are reported in ref. [9]. From those expressions, the N particle functions $j(\Gamma)$ and $\mathcal{M}(\Gamma)$, together with the eigenvalue λ appearing in Eq. (E5) for each of the transport coefficients, can be read off. Then, following the procedure illustrated above and, in the context of the shear viscosity in the main text, the results reported below are

obtained. Attention is restricted to the transport coefficients that were calculated in ref. [11], which excludes the transport coefficients ζ^T and ζ^n from the local cooling rate. Moreover, the parameter ℓ defining the length scale is chosen such that $n_h \ell^3 = 1$ in the following, for the sake of simplicity.

a. The pressure

The expression for the pressure has been identified in [9] as

$$p = 1 + \frac{(1 + \alpha) \sigma}{2Vd} \int dx_1 \int dx_2 \delta(q_{12} - \sigma) \Theta(-\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12}) (\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12})^2 f_h^{(2)}(\mathbf{q}_{12}, \mathbf{v}_1, \mathbf{v}_2). \quad (\text{E20})$$

This is the second moment of the normal component of the relative velocity averaged over the two-particle distribution at contact. In the Enskog approximation, all velocity correlations in the two-particle distribution are neglected, and the above expression simplifies to

$$p = 1 + \frac{\pi^{d/2}(1 + \alpha)\sigma^d}{2\Gamma(d/2)d} g_h^{(2)}(\sigma). \quad (\text{E21})$$

b. The zeroth order cooling rate ζ_0

The homogeneous dynamics in Eq. (E1) is determined entirely determined by ζ_0 , for which the expression

$$\zeta_0 = \frac{1 - \alpha^2}{2Vd} \int dx_1 \int dx_2 \delta(q_{12} - \sigma) \Theta(\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12}) |\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12}|^3 f_h^{(2)}(\mathbf{q}_{12}, \mathbf{v}_1, \mathbf{v}_2) \quad (\text{E22})$$

was derived in [9]. When the Enskog approximation for the two-particle distribution function is used, this simplifies to

$$\begin{aligned} \zeta_0 &= \frac{\pi^{(d-1)/2} \sigma^{d-1} g_h^{(2)}(\sigma) (1 - \alpha^2)}{2\Gamma\left(\frac{d+3}{2}\right) d} \int d\mathbf{v}_1 \int d\mathbf{v}_2 g_{12}^3 f_h^{(1)}(v_1) f_h^{(1)}(v_2) \\ &= \frac{\Gamma(d/2) (1 - \alpha)}{\pi^{1/2} \Gamma\left(\frac{d+3}{2}\right) \sigma} (p - 1) \int d\mathbf{v}_1 \int d\mathbf{v}_2 g_{12}^3 f_h^{(1)}(v_1) f_h^{(1)}(v_2). \end{aligned} \quad (\text{E23})$$

c. The Euler transport coefficient ζ^U

This is a new transport coefficient unique to granular fluids. Its expression, as obtained in ref. [9], reads

$$\zeta^U = \lim \Omega_H^{\zeta^U}(s) = \Omega_H^{\zeta^U}(0) + \lim \int_0^s ds' \Omega_G^{\zeta^U}(s'), \quad (\text{E24})$$

where

$$\Omega_H^{\zeta^U}(s) = V^{-1} \int d\Gamma W(\Gamma) (1 - \mathcal{P}) e^{-s(\bar{\mathcal{L}} + \frac{\zeta_0}{2})} \mathcal{M}_{\zeta^U}(\Gamma) \quad (\text{E25})$$

and

$$\Omega_G^{\zeta^U}(s) = V^{-1} \int d\Gamma W(\Gamma) (1 - \mathcal{P}) e^{-s(\bar{\mathcal{L}} + \frac{\zeta_0}{2})} \Upsilon_{\zeta^U}(\Gamma). \quad (\text{E26})$$

In the above expressions, $W(\Gamma)$ is the source term in the microscopic energy balance equation that characterizes the dissipation due to the inelastic collisions,

$$W(\Gamma) = \frac{1 - \alpha^2}{2d} \sum_{r=1}^N \sum_{s \neq r}^N \delta(q_{rs} - \sigma) \Theta(-\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}) |\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}|^3, \quad (\text{E27})$$

and the conjugate density and flux are

$$\mathcal{M}_{\zeta^U}(\Gamma) = - \sum_r^N \mathbf{q}_r \cdot \frac{\partial}{\partial \mathbf{v}_r} \rho_h(\Gamma), \quad (\text{E28})$$

$$\Upsilon_{\zeta^U}(\Gamma) = - \left(\bar{\mathcal{L}} + \frac{\zeta_0}{2} \right) \mathcal{M}_{\zeta^U}(\Gamma). \quad (\text{E29})$$

The Helfand form for this transport coefficient in the Enskog approximation can be obtained by application of Eq. (E9) with the result

$$\Omega_H^{\zeta^U}(s) \simeq V^{-1} \int dx J_{\zeta^U}(x) [1 - \mathcal{P}^{(1)}(x)] e^{-s(\Lambda^E + \frac{\zeta_0}{2})} \mathcal{M}_{\zeta^U}^{(1)}(x), \quad (\text{E30})$$

where

$$J_{\zeta^U}(\mathbf{v}) = \frac{\pi^{\frac{d-1}{2}} g_h^{(2)}(\sigma) \sigma^{d-1}}{\Gamma(\frac{d+3}{2}) d} \int d\mathbf{v}_1 |\mathbf{v} - \mathbf{v}_1|^3 f_h^{(1)}(v_1), \quad (\text{E31})$$

$$\mathcal{M}_{\zeta^U}^{(1)}(x_1) = -\mathbf{q}_1 \cdot \frac{\partial}{\partial \mathbf{v}_1} \tilde{\psi}_1^{(1)}(\mathbf{v}_1, \mathbf{0}), \quad (\text{E32})$$

and Λ^E is the collision operator given in Eq. (85). The Green-Kubo form of this transport coefficient is determined by $\Omega_H^{\zeta^U}(0)$ and $\Omega_G^{\zeta^U}(s)$. Direct evaluation of the former gives [9]

$$\Omega_H^{\zeta^U}(0) = -\frac{3}{d} (1 - \alpha) (p - 1). \quad (\text{E33})$$

Here p is the pressure in the Enskog approximation, Eq. (E21). The time correlation function is (see Eq. (E15))

$$\Omega_G^{\zeta^U}(s) = V^{-1} \int dx J_{\zeta^U}(x) [1 - \mathcal{P}^{(1)}(x)] e^{-s(\Lambda^E + \frac{\zeta_0}{2})} \gamma_{\zeta^U}(x) \quad (\text{E34})$$

with the conjugate flux given by

$$\gamma_{\zeta^U}(x) = - \left(\Lambda^E + \frac{\zeta_0}{2} \right) \mathcal{M}_{\zeta^U}^{(1)}(x). \quad (\text{E35})$$

Proceeding as in the case of the shear viscosity, it can be shown that

$$e^{-s(\Lambda^E + \frac{\zeta_0}{2})} \gamma_{\zeta^U}(\mathbf{v}) = e^{-s(\mathcal{J} + \frac{\zeta_0}{2})} \gamma_{\zeta^U}(\mathbf{v}), \quad (\text{E36})$$

where \mathcal{J} is the operator defined in Eq. (89). The time correlation function in Eq. (E34) can be expressed as

$$\Omega_G^{\zeta^U}(s) = \int d\mathbf{v} J_{\zeta^U}(\mathbf{v}) [1 - \mathcal{P}^{(1)}(x)] e^{-s(\mathcal{J} + \frac{\zeta_0}{2})} \gamma_{\zeta^U}(\mathbf{v}). \quad (\text{E37})$$

Therefore, this Euler order transport coefficient is given in the Enskog approximation by

$$\zeta^U = -\frac{3}{d}(1 - \alpha)(p - 1) + \int d\mathbf{v} J_{\zeta^U}(\mathbf{v}) [1 - \mathcal{P}^{(1)}(x)] \mathcal{D}_{\zeta^U}(\mathbf{v}), \quad (\text{E38})$$

where $\mathcal{D}_{\zeta^U}(\mathbf{v})$ is the solution to the integral equation

$$\left(\mathcal{J} + \frac{\zeta_0}{2}\right) \mathcal{D}_{\zeta^U}(\mathbf{v}) = [1 - \mathcal{P}^{(1)}(x)] \gamma_{\zeta^U}(\mathbf{v}). \quad (\text{E39})$$

d. The bulk viscosity κ

The expression for the bulk viscosity reported in ref. [9] is

$$\kappa = \lim \Omega_H^\kappa(s) = \Omega_H^\kappa(0) + \lim \int_0^s ds' \Omega_G^\kappa(s'), \quad (\text{E40})$$

where

$$\Omega_H^\kappa(s) = -\frac{1}{Vd^2} \int d\Gamma \text{tr} \mathbf{H} (1 - \mathcal{P}) e^{-s(\bar{\mathcal{L}} + \frac{\zeta_0}{2})} \mathcal{M}_\kappa(\Gamma), \quad (\text{E41})$$

and

$$\Omega_G^\kappa(s) = -\frac{1}{Vd^2} \int d\Gamma \text{tr} \mathbf{H} (1 - \mathcal{P}) e^{-s(\bar{\mathcal{L}} + \frac{\zeta_0}{2})} \Upsilon_\kappa(\Gamma). \quad (\text{E42})$$

The direct flux $\text{tr} \mathbf{H}$ is the trace of the momentum flux given in Eq. (66), and the adjoint density is the same as that for ζ^U , i.e.,

$$\mathcal{M}_\kappa(\Gamma) = \mathcal{M}_{\zeta^U}(\Gamma) = -\sum_{r=1}^N \mathbf{q}_r \cdot \frac{\partial}{\partial \mathbf{v}_r} \rho_h(\Gamma). \quad (\text{E43})$$

Consequently, the conjugate flux is also the same, $\Upsilon_\kappa = \Upsilon_{\zeta^U}$, given by (E29). In the Enskog approximation, the time correlation function $\Omega_H^\kappa(s)$ becomes

$$\Omega_H^\kappa(s) \simeq V^{-1} \int dx J_\kappa(x) [1 - \mathcal{P}^{(1)}(x)] e^{-s(\Lambda^E + \frac{\zeta_0}{2})} \mathcal{M}_\kappa^{(1)}(x), \quad (\text{E44})$$

with the direct flux given by

$$J_\kappa(x_1) = -\frac{v_1^2}{d^2} + \frac{1}{2d} \frac{\partial \ln g_h^{(2)}(\sigma)}{\partial n_h} (p-1) - \frac{(1+\alpha) \sigma g_h^{(2)}(\sigma)}{2d^2} \int dx_2 \delta(q_{12} - \sigma) \Theta(-\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12}) |\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12}|^2 f_h^{(1)}(v_2) \quad (\text{E45})$$

and $\mathcal{M}_\kappa^{(1)}(x) = \mathcal{M}_{\zeta^U}^{(1)}(x)$, given in Eq. (E32). The instantaneous contribution $\Omega_H^\kappa(0)$ is simply related to $\Omega_H^\eta(0)$ in (93) through [9]

$$\Omega_H^\kappa(0) = \frac{d+2}{d} \Omega_H^\eta(0). \quad (\text{E46})$$

Furthermore, the correlation function $\Omega_G^\kappa(s)$ vanishes. This can be seen as follows. The conjugate flux and the generator of dynamics can be simplified as in the case of ζ^U above, so that they become independent of the coordinate \mathbf{q} . Then, the direct flux above can be simplified to give

$$J_\kappa(\mathbf{v}_1) = - \left[1 + \frac{(1+\alpha) \sigma^{d/2} g_h^{(2)}(\sigma)}{2\Gamma(\frac{d}{2}) d} \right] \frac{v_1^2}{d^2} - \left[\frac{1}{2} \frac{\partial \ln g_h^{(2)}(\sigma)}{\partial n_h} + 1 \right] \frac{p-1}{d}. \quad (\text{E47})$$

Since $J_\kappa(\mathbf{v})$ is orthogonal to the subspace spanned by $1 - \mathcal{P}^{(1)}(x)$, $\Omega_G^\kappa(s)$ vanishes. Therefore, the bulk viscosity in the Enskog approximation is simply

$$\begin{aligned} \kappa &= \frac{d+2}{d} \Omega_H^\eta(0) = \frac{\pi^{(d-1)/2} (1+\alpha) \sigma^{d+1} g_h^{(2)}(\sigma)}{2d^2 \Gamma(\frac{d+1}{2})} \int d\mathbf{v}_1 \int d\mathbf{v}_2 |\mathbf{v}_1 - \mathbf{v}_2| f_h^{(1)}(v_1) f_h^{(1)}(v_2) \\ &= \frac{\pi^{-1/2} \Gamma(d/2) \sigma (p-1)}{\Gamma(\frac{d+1}{2}) d} \int d\mathbf{v}_1 \int d\mathbf{v}_2 |\mathbf{v}_1 - \mathbf{v}_2| f_h^{(1)}(v_1) f_h^{(1)}(v_2). \end{aligned} \quad (\text{E48})$$

e. The thermal conductivity λ

The thermal conductivity is expressed in ref. [9] as

$$\lambda = \lim \Omega_H^\lambda(s) = \Omega_H^\lambda(0) + \lim \int_0^s ds' \Omega_G^\lambda(s'), \quad (\text{E49})$$

where

$$\Omega_H^\lambda(s) = -(Vd)^{-1} \int d\Gamma \mathbf{S}(\Gamma) \cdot (1 - \mathcal{P}) e^{-s(\bar{\mathcal{L}} - \frac{\zeta_0}{2})} \mathbf{M}_\lambda(\Gamma) \quad (\text{E50})$$

and

$$\Omega_G^\lambda(s) = -(Vd)^{-1} \int d\Gamma \mathbf{S}(\Gamma) \cdot (1 - \mathcal{P}) e^{-s(\bar{\mathcal{L}} - \frac{\zeta_0}{2})} \mathbf{r}_\lambda(\Gamma). \quad (\text{E51})$$

The direct flux \mathbf{S} is the heat flux in the microscopic balance equation for the energy density,

$$\mathbf{S} = \sum_{r=1}^N v_r^2 \mathbf{v}_r + \sum_{r=1}^N \sum_{s \neq r}^N \mathbf{s}_2(x_r, x_s), \quad (\text{E52})$$

with

$$\mathbf{s}_2(x_r, x_s) = \frac{(1+\alpha)\sigma}{2} \delta(q_{rs} - \sigma) \Theta(-\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs}) (\hat{\mathbf{q}}_{rs} \cdot \mathbf{g}_{rs})^2 (\hat{\mathbf{q}}_{rs} \cdot \mathbf{G}_{rs}) \hat{\mathbf{q}}_{rs}, \quad (\text{E53})$$

where $\mathbf{G}_{rs} \equiv (\mathbf{r} + \mathbf{v}_s)/2$. The conjugate density in Eq. (E50) is

$$\mathcal{M}_\lambda(\Gamma) = -\frac{1}{2} \sum_{r=1}^N \mathbf{q}_r \frac{\partial}{\partial \mathbf{v}_r} \cdot [\mathbf{v}_r \rho_h(\Gamma)] \quad (\text{E54})$$

and the associated flux is

$$\Upsilon_\lambda(\Gamma) = -\left(\bar{\mathcal{L}} - \frac{\zeta_0}{2}\right) \mathcal{M}_\lambda(\Gamma). \quad (\text{E55})$$

In the Enskog approximation, the time correlation function determining the Helfand form of the thermal conductivity λ is

$$\Omega_H^\lambda(s) \simeq V^{-1} \int dx \mathbf{J}_\lambda(x) \cdot [1 - \mathcal{P}^{(1)}(x)] e^{-s(\Lambda^E - \frac{\zeta_0}{2})} \mathcal{M}_\lambda^{(1)}(x), \quad (\text{E56})$$

where

$$\begin{aligned} \mathbf{J}_\lambda(x_1) = & \frac{v_1^2 \mathbf{v}_1}{d} - \frac{(1+\alpha)\sigma g_h^{(2)}(\sigma)}{d} \int dx_2 \delta(q_{12} - \sigma) \Theta(-\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12}) \\ & \times (\hat{\mathbf{q}}_{12} \cdot \mathbf{g}_{12})^2 (\hat{\mathbf{q}}_{12} \cdot \mathbf{G}_{12}) \hat{\mathbf{q}}_{12} f_h^{(1)}(v_2) \end{aligned} \quad (\text{E57})$$

and

$$\mathcal{M}_\lambda^{(1)}(x_1) = -\frac{\mathbf{q}_1}{2} \frac{\partial}{\partial \mathbf{v}_1} \cdot [\mathbf{v}_1 f_h^{(1)}(v_1)]. \quad (\text{E58})$$

The function J_λ defined in Eq. (E57) differs from the generic form given in (E10) by a velocity independent term which does not contribute to Ω_H^λ . The contribution from the initial correlation function to the Green-Kubo form of this transport coefficient is given by

$$\begin{aligned} \Omega_H^\lambda(0) = & \frac{\Gamma(\frac{d}{2})}{2\pi^{1/2}\Gamma(\frac{3+d}{2})} \sigma(p-1) \int d\mathbf{v}_1 \int d\mathbf{v}_2 f_h^{(1)}(v_1) f_h^{(2)}(v_2) \\ & \times \left[g_{12} G_{12}^2 + g_{12} (\hat{\mathbf{g}}_{12} \cdot \mathbf{G}_{12})^2 + \frac{1}{4} g_{12}^3 + \frac{3}{2} (\hat{\mathbf{g}}_{12} \cdot \mathbf{G}_{12}) \right] \end{aligned} \quad (\text{E59})$$

while the time-dependent correlation function becomes

$$\Omega_G^\lambda(s) = -\left[1 + \frac{3(p-1)}{d+2}\right] \int d\mathbf{v} \frac{v^2 \mathbf{v}}{d} \cdot [1 - \mathcal{P}^{(1)}(\mathbf{v})] e^{-s(\mathcal{J} - \frac{\zeta_0}{2})} \gamma_\lambda(\mathbf{v}). \quad (\text{E60})$$

Proceeding as in the case of the shear viscosity in Sec. VII, it is found that

$$\gamma_\lambda = \left(\Lambda^E - \frac{\zeta_0}{2} \right) \frac{1}{2} \mathbf{q} \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} f_h^1(v)]. \quad (\text{E61})$$

Thus the thermal conductivity in the Enskog approximation is given by

$$\lambda = \Omega_H^\lambda(0) - \left[1 + \frac{3(p-1)}{d+2} \right] \int d\mathbf{v} \frac{v^2 \mathbf{v}}{d} \cdot [1 - \mathcal{P}^{(1)}(\mathbf{v})] \mathcal{A}(\mathbf{v}), \quad (\text{E62})$$

where \mathcal{A} is the solution to the integral equation

$$\left(\mathcal{J} - \frac{\zeta_0}{2} \right) \mathcal{A}(\mathbf{v}) = [1 - \mathcal{P}^{(1)}(\mathbf{v})] \gamma_\lambda(\mathbf{v}), \quad (\text{E63})$$

with \mathcal{J} the operator defined in Eq. (89).

f. The coefficient μ

This is a new transport mechanism for granular fluids that arises due to the inelasticity of collisions. As discussed in detail in ref. [9], this transport coefficient consists of two time correlation functions, one of which can be recognized as the time correlation part of the thermal conductivity. Therefore, the quantity in terms of which the time correlation function takes the simplest form is the linear combination

$$\bar{\mu} \equiv \mu - 2 \frac{\partial \ln \zeta_h}{\partial \ln n_h} \lambda = \lim \Omega_H^{\bar{\mu}}(s) = \Omega_H^{\bar{\mu}}(0) + \lim \int_0^s ds' \Omega_G^{\bar{\mu}}(s'), \quad (\text{E64})$$

with

$$\Omega_H^{\bar{\mu}}(s) = -(Vd)^{-1} \int d\Gamma \mathbf{S}(\Gamma) \cdot (1 - \mathcal{P}) e^{-s\bar{\mathcal{L}}} \mathcal{M}_{\bar{\mu}}(\Gamma), \quad (\text{E65})$$

and

$$\Omega_G^{\bar{\mu}}(s) = -(Vd)^{-1} \int d\Gamma \mathbf{S}(\Gamma) \cdot (1 - \mathcal{P}) e^{-s\bar{\mathcal{L}}} \Upsilon_\mu(\Gamma). \quad (\text{E66})$$

The direct flux \mathbf{S} above is the heat flux given in Eq. (E52). The conjugate density is

$$\begin{aligned} \mathcal{M}_{\bar{\mu}} &= \int d\mathbf{r} \mathbf{r} \left[\frac{\delta \rho_{lh}}{\delta n(\mathbf{r})} - 2 \frac{\partial \ln \zeta_0}{\partial \ln n} T \frac{\delta \rho_{lh}}{\delta T(\mathbf{r})} \right]_{y=y_h} \\ &= \int d\mathbf{r} \mathbf{r} \left[\left(\frac{\delta \rho_{lh}[\Gamma|y]}{\delta n(\mathbf{r})} \right)_{\zeta_0} \right]_{y=y_h}, \end{aligned} \quad (\text{E67})$$

where the functional derivative with respect to the density in the last equality is to be taken at constant cooling rate, as indicated. In the Enskog approximation, the Helfand form becomes

$$\Omega_H^{\bar{\mu}}(s) \simeq V^{-1} \int dx \mathbf{J}_{\bar{\mu}}(x) \cdot [1 - \mathcal{P}^{(1)}(x)] e^{-s(\Lambda^E - \frac{\zeta_0}{2})} \mathcal{M}_{\bar{\mu}}^{(1)}(x). \quad (\text{E68})$$

Here, $\mathbf{J}_{\bar{\mu}}(x) = \mathbf{J}_{\lambda}(x)$, given in Eq. (E57), and

$$\mathcal{M}_{\bar{\mu}}^{(1)} = \mathbf{q} \left\{ f_h^{(1)}(v) + \frac{\partial \ln \zeta_0}{\partial \ln n_h} \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} f_h^{(1)}(v)] \right\}. \quad (\text{E69})$$

The Green-Kubo form is determined from $\Omega_H^{\bar{\mu}}(0)$ and $\Omega_G^{\bar{\mu}}(s)$. Direct evaluation shows the relationship

$$\Omega_H^{\bar{\mu}}(0) = -2 \frac{\partial \ln \zeta_0}{\partial \ln n_h} \Omega_H^{\lambda}(0). \quad (\text{E70})$$

The time correlation function $\Omega_G^{\bar{\mu}}(s)$ can be simplified as in the previous cases yielding

$$\Omega_G^{\mu}(s) = \int d\mathbf{v} \mathbf{J}_{\bar{\mu}}(\mathbf{v}) \cdot [1 - \mathcal{P}^{(1)}(\mathbf{v})] e^{-s\mathcal{J}} \gamma_{\bar{\mu}}(\mathbf{v}), \quad (\text{E71})$$

with

$$\gamma_{\mu,i}(\mathbf{v}) = -v_i f_h^{(1)}(v) - \left[1 + \frac{1}{2} \frac{\partial \ln g_h^{(2)}(\sigma)}{\partial \ln n_h} \right] Q_i [f_h^{(1)}(v)] - 2 \frac{\partial \ln \zeta_0}{\partial \ln n_h} \gamma_{\lambda,i}. \quad (\text{E72})$$

The Green-Kubo form for the transport coefficient μ is

$$\mu - 2 \frac{\partial \ln \zeta_0}{\partial \ln n_h} \lambda = -2 \frac{\partial \ln \zeta_0}{\partial \ln n_h} \Omega_H^{\lambda}(0) + \int d\mathbf{v} \mathbf{J}_{\lambda}(\mathbf{v}) \cdot [1 - \mathcal{P}^{(1)}(x)] \mathbf{C}(\mathbf{v}), \quad (\text{E73})$$

where $\mathbf{C}(\mathbf{v})$ is a solution to the integral equation

$$\mathcal{J}\mathbf{C}(\mathbf{v}) = [1 - \mathcal{P}^{(1)}(x)] \mathbf{J}_{\lambda}(\mathbf{v}). \quad (\text{E74})$$

All of the above results agree in detail (for $d = 3$) with those obtained in ref. [11] by means of the Chapman-Enskog solution of the non-linear Enskog kinetic equation.

-
- [1] See, for example, P.C. Martin in *Many Body Physics*, edited by C. de Witt and R. Balian (Gordon and Breach, New York, 1968).
 - [2] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading MA, 1975).
 - [3] J-P. Boon and S. Yip, *Molecular Hydrodynamics* (Dover, New York, 1991).
 - [4] J.A. McLennan, *Introduction to Nonequilibrium Statistical Mechanics* (Prentice-Hall, New Jersey, 1989).
 - [5] E. Helfand, Phys. Rev. **119**, 1 (1960).

- [6] J.A. McLennan, in *Advances in Chemical Physics*, edited by I. Prigogine (Interscience Publishers, Inc., New York, 1963), Vol. 5.
- [7] P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (Wiley Interscience, New York, 1977).
- [8] J.W. Dufty, A. Baskaran, and J.J. Brey, submitted to Phys. Rev. E.,cond-mat/0612408
- [9] A. Baskaran, J.W. Dufty, and J.J. Brey, submitted to Phys. Rev. E.,cond-mat/0612409
- [10] Hydrodynamics derived from inelastic hard sphere models has found widespread use in multi-phase computational fluid dynamics methods, to describe numerous industrial processes involving solid particulates in gas fluidized systems (e.g., software packages such as Fluent 2122 and MFIx).
- [11] V. Garzó and J.W. Dufty, Phys. Rev. E **59**, 5895 (1999).
- [12] J.W. Dufty, J.J. Brey, and J. Lutsko, Phys. Rev. E **65**, 051303 (2002); J. Lutsko, J.J. Brey, and J.W. Dufty, Phys. Rev. E **65**, 051304 (2002).
- [13] V. Garzó and J.W. Dufty, J. Stat. Phys. **105**, 723-744 (2001).
- [14] N. N. Bogoliubov, *Problems in a Dynamical Theory in Statistical Physics*, English trans. by E. Gora in *Studies in Statistical Mechanics I*, edited by G.E. Uhlenbeck and J. de Boer (North-Holland, Amsterdam, 1962).
- [15] E.G.D Cohen, Physica **28**, 1025 (1962); *ibid.* **28**, 1045 (1962); *ibid.* **28** 1060 (1962).
- [16] Y. Pomeau and P. Résibois, Phys. Rep. **19**, 63 (1975); J. R. Dorfman in *Perspectives in Statistical Physics*, edited by H. Raveche (North-Holland, Amsterdam, 1981).
- [17] B. Alder and T. Wainwright, Phys. Rev. A **1**, 18 (1970).
- [18] E. Leutheusser, Phys. Rev. A **29**, 2765 (1984).
- [19] T. P. C. van Noije and M. H. Ernst, in *Granular Gases*, edited by T. Pöschel and S. Luding (Springer, New York, 2001).
- [20] H. van Beijeren and M.H. Ernst, Physica A **68**, 437 (1973); *ibid.* **70**, 225 (1973); J. Stat. Phys. **21**, 125 (1979).
- [21] W.E. Alley, B.J. Alder, and S. Yip, Phys. Rev. A **27**, 3174 (1984).
- [22] I.M. de Schepper, E.G.D. Cohen, C. Bruin, J.C. van Rijs, W. Montfrooij, and L.A. de Graaf, Phys. Rev. A **38**, 271 (1988).
- [23] J.F. Lutsko, Phys. Rev. E **63**, 061211 (2001).
- [24] J. W. Dufty and J.J. Brey, Phys. Rev. E **68**, 030302(R) (2003); J.J. Brey and J.W. Dufty,

- Phys. Rev. E **72**, 011303 (2005).
- [25] J.J. Brey, M.I. García de Soria, P. Maynar, and M.J. Ruiz-Montero, Phys. Rev. E **70**, 051301 (2004).
- [26] J.J. Brey, M.J. Ruiz-Montero, P. Maynar, and M.I. García de Soria, J. Phys.: Condens. Matter **17**, S2489 (2005).
- [27] J-P. Hansen and I. R. McDonald, *Theory of Simple Liquids*, (Elsevier Academic Press, London, 1986).